S E L E C T E D T O P I C S

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G R O U P T H E O R Y

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F O R E W O R D

These notes are based on lectures given by one of the authors (N.S.) at Duke University in 1968/69. They contain some of the grouptheoretical tools which have turned out to be useful in many branches of physics. We have tried to develop these tools in as self-contained a manner as possible, using a reasonably modern mathematical language. This should facilitate the reader to consult also books written mainly for mathematicians.

One of the authors (N.S.) had the privilege of many informative discussions on the subject with Professor L.C. Biedenharn. We are also indepted to L.Weaver who has cleaned up a first version of the manuscript.

T A B L E   O F   C O N T E N T S

CHAPTER I SEMISIMPLE ASSOCIATIVE ALGEBRAS

1. Fundamental Concepts of Algebra

We begin with a series of definitions which play an important role in the following chapters.

Definition 1 A semigroup is a system (S, .), where S is a set and . is a binary operation on S (mapping from S x S into S) which satisfies the associative law: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
We shall usually omit the . and write simply ab instead of a . b. A semigroup with 1 (often called a "monoid") is a system (S, 1, .), where (S, .) is a semigroup and 1 is a designated element of S satisfying the identity: $1\,a = a = a\,1$.

Definition 2 A group is a system $(S, 1, {}^{-1}, .)$, where (S, 1, .) is a semigroup with 1 and $^{-1}$ is a unary operation ($^{-1}$ maps $S \to S$) such that $a\,a^{-1} = 1 = a^{-1}a$.
An abelian group is a group satisfying the commutative law $a\,b = b\,a$. An abelian group is frequently written as (S, 0, -, +).

Definition 3 A ring (associative with 1) is a system (S, 0, 1, -, +, .), where (S, 0, -, +) is an abelian group and (S, 1, •) is a semigroup with 1, satisfying the distributive laws $a\,(b+c) = a\,b + a\,c$, $(a+b)\,c = a\,c + b\,c$.
A ring in which every element $a \neq 0$ is invertible (relative to multiplication) and which contains at least two elements is called a field.

Definition 4 A right R-module M_R consists of an additive abelian group M, a ring R, and a mapping $M \times R \longrightarrow M$, denoted by juxtaposition, such that

$(m_1 + m_2)r = m_1 r + m_2 r$, $m(r_1 + r_2) = m r_1 + m r_2$,

$m(r_1 r_2) = (m r_1) r_2$, $m 1 = m$

for all $m_1, m_2 \in M$ and $r_1, r_2 \in R$. A left R-module $_R M$ is defined symmetrically.

Examples:

1. If R is a field, M_R is usually called a vector space.
2. If M = R and the mapping $M \times R \longrightarrow M$ is taken to be multiplication, that is $m\ r = m\ .\ r$, then we have the right module R_R.

Definition 5 An algebra A over a field ϕ is a ring A which is at the same time a vector space over ϕ. Moreover the scalar multiplication in the vector space and the ring multiplication are required to satisfy the axiom

$\alpha(a b) = (\alpha a) b = a(\alpha b)$, $\alpha \in \phi$, a, b $\in$ A.

Definition 6 Let A be an algebra over ϕ, and let M be a vector space over ϕ. We say that M is a left A-module (or a module over A) if (i) M is a left A-module, A considered as a ring and (ii) $(\alpha a) m = \alpha (am) = a(\alpha m)$ for all $\alpha \in \phi$, $a \in A$, $m \in M$.

Let M, N be R-modules. We use the notation $Hom_R(M,N)$ to denote the set of all <u>R-homomorphisms</u> of M into N, that is, the set of all mappings $f: M \longrightarrow N$ such that $f(m_1 + m_2) = f(m_1) + f(m_2)$, $f(rm) = r\ f(m)$; $m_i \in M$, $r \in R$.
The set $Hom_R(M, N)$ forms a subgroup of the set Hom (M, N) (homomorphisms of M into N, M and N considered as abelian groups), and $Hom_R(M, M)$ forms a subring of Hom (M, M). We call $Hom_R(M, M)$ the <u>ring of R-endomorphisms of M</u>, or sometimes the <u>centralizer</u> of the R-module M, because the elements of $Hom_R(M, M)$ are precisely those endomorphisms f of M which commute with all the endomorphisms of M $r_L : m \longrightarrow rm$ determined by the elements of R. Similarly one defines the algebra of A-endomorphisms of a module M over an algebra.

Definition 7 Let A be an algebra over a field ϕ and M a vector space over ϕ. A representation of A with representation space M is an algebra homomorphism $T: A \longrightarrow \mathrm{Hom}_{\phi}(M, M)$, that is, a mapping T which satisfies $T(a + b) = T(a) + T(b)$, $T(ab) = T(a)\,T(b)$ $T(\alpha a) = \alpha T(a)$, $T(e) = 1$, $a, b \in A$, $\alpha \in \phi$, where e is the identity element of A.

Let $T: A \longrightarrow \mathrm{Hom}_{\phi}(M, M)$ be a representation of an algebra A over ϕ. Then, for each $a \in A$ and $m \in M$, we define

$$a\,m = T(a)\,m \qquad (1)$$

and observe that, because of the properties of the representation T, we have for all $a, a' \in A$, $m, m' \in M$, $\alpha \in \phi$,

$$a(m + m') = a\,m + am', \quad (a + a')\,m = a\,m + a'\,m \qquad (2)$$
$$(a\,a')\,m = a\,(a'm), \quad em = m, \quad (\alpha\,a)\,m = \alpha\,(a\,m) = a\,(\alpha\,m)$$

where e is the identity element of A. It is clear that the definition (1) turns M into an A-module. Conversely, let M be a ϕ - space (vector space over ϕ) which is a left A-module for an algebra over ϕ. For each $a \in A$, define a mapping $T(a): M \longrightarrow M$ by setting $T(a)\,m = a\,m$, $a \in A$, $m \in M$. Then $T(a) \in \mathrm{Hom}_{\phi}(M, M)$ for each $a \in A$. Moreover $a \longrightarrow T(a)$ is a representation of A. So representations and modules are essentially the same. We shall in the following usually state and prove the theorems in the language of modules.

Definition 8 Let M and M' be left A-modules where A is an algebra over a field ϕ. The modules M and M' are said to be A-isomorphic if there exists a vector-space isomorphism S of M onto M' such that for all $a \in A$ and $m \in M$ we have $a(S\,m) = S(a\,m)$. Clearly, two modules are A-isomorphic if and only if the representations afforded by them are equivalent.

Definition 9 Let M be a left A-module over a ϕ-algebra A where ϕ is a field. A ϕ-subspace N of M is called a submodule if $an \in N$ for all $a \in A$ and $n \in N$. For example, the submodules of the left regular module ${}_{A}A$ are the left ideals of A.

2. Direct Sums of Modules

If M_1 and M_2 are submodules of the R-module M (let all R-modules be left modules), we define the sum of M_1 and M_2 by
$M_1 + M_2 = \{m_1 + m_2 : m_1 \in M_1, m_2 \in M_2\}$.
Then $M_1 + M_2$ is again a submodule of M and is the smallest submodule which contains both M_1 and M_2. The intersection $M_1 \cap M_2$ is the largest submodule contained in both M_1 and M_2.

Now let $M_1, \dots M_k$ be submodules of the R-module M. We write $M = M_1 \oplus \dots \oplus M_k$ and call M the (internal) direct sum of $M_1, \dots M_k$ if

(i) $M = M_1 + \dots + M_k$

(ii) $m_1 + \dots + m_k = 0$, $m_i \in M_i$, implies that each $m_i = 0$.

It is easily verified that if (i) holds then (ii) is equivalent to either of the following two conditions:

(ii') $M_i \cap (M_i + \dots + M_{i-1} + M_{i+1} + \dots + M_k) = 0$ for each i.

(ii'') Every element $m \in M$ can be expressed uniquely as a sum $m = m_1 + \dots + m_k$, $m_i \in M_i$.

Now let $M_1, \dots M_k$ be a given set of R-modules, not necessarily submodules of a common R-module. We define their (external) direct sum $M_1 \oplus \dots \oplus M_k = M^*$ to be the set of k-tuples $(m_1, \dots, m_k)$, $m_i \in M_i$, where addition is defined componentwise, and $r(m_1, \dots, m_k) = (r m_1, \dots, rm_k)$, $r \in R$. Then M^* is an R-module; if we set $M'_i = \{(0, \dots, m_i, 0 \dots, 0); m_i \in M_i\}$, then M'_i is a submodule of M^*, and $M'_i \cong M_i$. Moreover M^* is the internal direct sum of the submodules M'_i.

If N is a submodule of M, the factor module M/N is the left R-module whose underlying commutative group is the totality of cosets $\{m + N\}$ of N in M, and the module composition is defined by

$$r(m + N) = r m + N; \qquad (3)$$

it is well defined because N is a submodule. The mapping π: $m \rightarrow m + N$, which maps $m \in M$ onto the coset containing it, is by (3) an R-homomorphism of M onto M/N, called the natural homomorphism of M onto M/N.

3. Classical Isomorphism Theorems

For completeness we also prove the classical isomorphism theorems.

Proposition 1

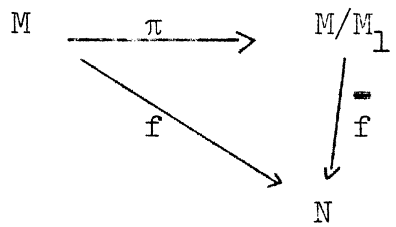
Let M, N be R-modules, and let $f: M \rightarrow N$ be an R-homomorphism of M onto N. Let $M_1 = f^{-1}(0)$ be the kernel of f; then M_1 is a submodule of M, and $M/M_1 \cong N$.

Proof:

M_1 is obviously a submodule of M. Let π be the natural mapping of M onto M/M_1 and define (see the diagram)

$\bar{f}: M/M_1 \rightarrow N$ by

$\bar{f}: \pi(m) \rightarrow f(m)$



(4) , $\bar{f} \circ \pi = f$

This is a well defined mapping since the elements of any coset $m + M_1$ are mapped by f to a single element. $\bar{f}$ is a R-homomorphism. For instance

$\bar{f}(r(m + M_1)) = f(rm) = r f(m) = r \bar{f}(m + M_1)$.

Proposition 2

Let N be a submodule of the R-module M. Every submodule of M/N has the form L/N where $N \subset L \subset M$, and $M/L \cong (M/N)/(L/N)$.

Proof:

Let π: M $\longrightarrow$ M/N the canonical epimorphism (homomorphism onto). Any submodule I of $\pi(M)$ = M/N has an inverse image L = $\pi^{-1}(I)$ in M, hence I = $\pi(L)$. Clearly L is a submodule of M. Furthermore N = $\pi^{-1}(0) \subset L$, and so we may write $\pi(L)$ = L/N. Now let $\pi'(M) \longrightarrow \pi(M)/\pi(L)$ be the canonical eipmorphism, then $\pi' \circ \pi$: M $\longrightarrow \pi(M)/\pi(L)$, and the kernel of this mapping is $(\pi' \circ \pi)^{-1}(0) = \pi^{-1}(\pi(L)) = L$. The result now follows from Proposition 1.

Proposition 3

Let M, N be submodules of a common R-module. Then $(M + N)/M \cong N/M \cap N$.

Proof:

Consider the canonical epimorphism π: M + N $\longrightarrow$ (M + N)/M and the monomorphism (injective homomorphism) χ : N $\longrightarrow$ M + N. Then $\pi \circ \chi$ has kernel $M \cap N$ and image $\pi\chi(N) = \pi(M) + \pi(N) = \pi(M + N) = (M + N)/M$. The result now follows from Proposition 1.

Definition 10

An R-module M is said to be indecomposable if M $\neq$ (0) and if it is impossible to express M as a direct sum of two non-trivial submodules.

Remark:

Since the left ideals of a ring R are the submodules of the module $_RR$ we say correspondingly that a left ideal is indecomposable if it is impossible to express it as a direct sum of two non-trivial left ideals.

Definition 11

A left R-module M $\neq$ (0) is called irreducible if M contains no non-trivial submodules, whereas a module which contains a non-trivial submodule is called reducible.

Definition 12

A left R-module M is said to be completely reducible if every submodule is a direct summand; in other words, for every sub-module N there exists a submodule N' such that $M = N \oplus N'$.

The following theorem is important.

Theorem 1.

The following two statements about an R-module M are equivalent

(i) M is completely reducible

(ii) M is a direct sum of irreducible modules.

Proof:

We prove this theorem only for finite dimensional A-modules, where A is an algebra (this is all we need in the following). Assume $M = \bigoplus_i M_i$, M_i irreducible modules. Let N be a submodule of M. If dim N = dim M, N = M and $M = N \oplus (0)$. We now suppose dim N < dim M and we may assume the theorem for submodules N_1 such that $\dim N_1 > \dim N$. Since $N \subset M = \bigoplus_i M_i$ there is an M_i such that $M_i \not\subseteq N$. Consider the submodule $M_i \cap N$. This is a submodule of the irreducible module M_i. Hence either $M_i \cap N = M_i$ or $M_i \cap N = 0$. If $M_i \cap N = M_i$, $N \supseteq M_i$ contrary to assumption. Hence $M_i \cap N = 0$ and $N_1 \equiv N + M_i = N \oplus M_i$. We can now apply the induction hypothesis to conclude that $M = N_1 \oplus N'_1$, where N'_1 is a submodule. Then $M = N \oplus M_i \oplus N'_1 = N \oplus N'$ where $N' = M_i \oplus N'_1$.

Conversely assume that M is completely reducible. Let M_1 be a submodule $\neq 0$ of minimal dimension. Then we have $M = M_1 \oplus N$ where N is a submodule. We note now that the condition assumed for M carries over to N. Thus let P be a submodule of N. Then we can write $M = P \oplus P'$, P' a submodule. Then by Dedekind's modular law *), (we denote for a moment the sum of two vectorspaces $M_1, M_2 \subseteq M$ by $M_1 + M_2 = M_1 \cup M_2$), $N = M \cap N = (P \oplus P') \cap N = P \cup (P' \cap N)$. Since $P \cap (P' \cap N) \subset P \cap P' = 0$

*) see next page.

we have $N = P \oplus P''$ with $P'' = P' \cap N$. We can now repeat for N the step taken for M. Continuing in this way we obtain, because of the finite dimensionality that $M = M_1 \oplus \dots \oplus M_n$. This completes the proof.

*) Dedkind's modular law says: If $N_1 \subseteq N_3$ then $N_1 \cup (N_2 \cap N_3) = (N_1 \cup N_2) \cap N_3$.

Proof:

$N_1 \subseteq N_3$ and $N_2 \cap N_3 \subseteq N_2 \subseteq N_1 \cup N_2$ implies $N_1 \cup (N_2 \cap N_3) \subseteq (N_1 \cup N_2) \cap N_3$. On the other hand let $x \in N_3 \cap (N_1 \cup N_2)$, that is $x = x_1 + x_2$ with $x_i \in N_i$. $x_2 = x - x_1 \in N_3$ since $N_1 \subseteq N_3$, hence $x_2 \in N_2 \cap N_3$. This implies $x \in N_1 \cup (N_2 \cap N_3)$, q.e.d.

Next we show that the decomposition of a completely reducible finite-dimensional A-module M into irreducible submodules is essentially unique. To prove this we need the following facts.

Lemma 1: Let M be a completely reducible A-module

$$M = M_1 \oplus \dots \oplus M_n, \quad M_i \text{ irreducible.}$$

Let N be a submodule of M. Then M is the direct sum of N and some of the M_i.

Proof: Consider $N \cap M_1$. Since M_1 is irreducible $M_1 \cap N = 0$ or $M_1 \cap N = M_1$ and hence $M_1 \subseteq N$. In the first case $N_1 \equiv N + M_1 = N \oplus M_1$. Now we consider $N_1 \cap M_2$ and repeat the same argument. In this way we obtain $M = N \oplus M_{i_1} \oplus \dots \oplus M_{i_u}$, where the M_{i_k} are those for which the first case in the above alternative happens to be true.

Lemma 2: Suppose that an A-module M can be decomposed in the following two ways:

$$M = M_1 \oplus M_2 = M_1 \oplus M'_2$$

where M_1, M_2, and M'_2 are submodules. Then $M_2 \cong M'_2$.

Proof: Consider the projection $\pi'_2: M \to M'_2$ which is defined by the second decomposition. The kernel of this homomorphism is M_1, hence $M_2 \cong M/M_1 \cong M'_2$.

Now we prove

Theorem 2.

Let M be an A-module and let $M = M_1 \oplus \ldots \oplus M_m = N_1 \oplus \ldots \oplus N_n$ be two decompositions of M into direct sums of irreducible submodules. Then $m = n$, and there exists a permutation $\{ j_1, \ldots j_n \}$ of $\{ 1, \ldots, n \}$ such that

$$M_1 \cong N_{j1}, \ldots, M_n \cong N_{jn}.$$

Proof: Assume $m \leq n$. We show a little bit more: For a suitable relabeling of the N_j,

$N_j \cong M_j$, $j = 1, \ldots m$ and $M = N_1 \oplus \ldots \oplus N_j \oplus M_{j+1} \oplus \ldots \oplus M_m$ for $j = 1, \ldots, m$. For $j = m$ this implies $m = n$. (5)

Equation (5) is true for $j = 1$. We suppose, that we already know, that after a suitable relabeling of the N_j, $N_j \cong M_j$ for $j = 1, \ldots, k-1$ and that

$$M = N_1 \oplus \ldots \oplus N_{k-1} \oplus M_k \oplus \ldots \oplus M_m \qquad (6)$$

Now we apply lemma 1 with $N = N_1 \oplus \ldots \oplus N_{k-1} \oplus M_{k+1} \oplus \ldots \oplus M_m$ and the second decomposition $M = N_1 \oplus \ldots \oplus N_k$ to conclude that M is the direct sum of N and some of the N_j with $j > k - 1$

$$M = N_1 \oplus \ldots \oplus N_{k-1} \oplus \Big(\sideset{}{'}\bigoplus_{j > k-1} N_j \Big) \oplus M_{k+1} \oplus \ldots \oplus M_m$$

Comparison with the decomposition (6) shows, using lemma 2, that M_k is isomorphic to $\sideset{}{'}\bigoplus_{j>k-1} N_j$. Hence this sum is irreducible and contains only one term which we label by k. This proves that (5) holds for $j = k$ and by induction our statement is proven.

4. The Radical of a Finite-Dimensional Algebra

If B_1 and B_2 are subspaces of an algebra A, then we write $B_1 \cap B_2$ and $B_1 + B_2$, respectively, for intersection and space spanned by B_1 and B_2. The latter is just the collection of elements of the form $b_1 + b_2$, $b_i \in B_i$. We define $B_1 . B_2$ to be the subspace spanned by all products $b_1 b_2$, $b_i \in B_i$. It is immediate that this is the set of (finite) sums

$$\sum_j b_{1j} b_{2j}, \; b_{ij} \in B_i .$$

It is trivial to verify the following equalities for subspaces

$$\text{(i)} \quad B_1 (B_2 B_3) = (B_1 B_2) B_3$$

$$\text{(ii)} \quad B_1 (B_2 + B_3) = B_1 B_2 + B_1 B_3$$

$$\text{(iii)} \quad (B_2 + B_3) B_1 = B_2 B_1 + B_3 B_1 \quad .$$

A subspace B is a <u>left-ideal</u> of A if and only if $AB \subseteq B$. Clearly the intersection and sum of two ideals is an ideal and (i) for $B_3 = A$ shows that the same is true of the product of ideals (ideal means left, right, or two-sided ideal). An ideal N is called <u>nilpotent</u> in case there exists an integer k such that $N^k = 0$.

<u>Proposition 4</u>

The sum of any finite number of nilpotent left ideals is nilpotent.

Proof: Let N_1 and N_2 be nilpotent left ideals in A; the sum $N_1 + N_2$ is also a left ideal. Let $N_1{}^k = N_2{}^r = 0$. Then every element in $(N_1 + N_2)^{k+r}$ is a sum of products $x_1 \ldots x_{k+r}$ in which either at least k factors belong to N_1 or at least r factors belong to N_2. In the former case, the above product may be written as

$$(x_1 \ldots x_{i_1}) \; (x_{i_1+1} \ldots x_{i_2}) \ldots (x_{i_{s-1}+1} \ldots x_{i_s}) \ldots ,$$

where $x_{i_1}, x_{i_2}, \ldots x_{i_s} \in N_1$ and $s \geqslant k$. Each group in parentheses belongs to N_1 since N_1 is a left ideal. However, the product of any s elements of N_1 is 0, and so the above product is 0. A similar argument holds when at least r factors belong to N_2. This completes the proof.

From now on, the algebra A is assumed to be <u>finite dimensional.</u>

<u>Proposition 5</u>

In A there exists a maximal nilpotent left ideal N. The ideal N is a two-sided ideal and contains every nilpotent left ideal and every nilpotent right ideal.

Proof: Let N be a nilpotent left ideal of maximal dimension. Assume that a nilpotent left ideal N_1 is not contained in N. Then $N_1 + N \supsetneq N$ is according to Proposition 4 a nilpotent left ideal which contains N. This is in contradiction to the assumption. Now NA is also a nilpotent left ideal since

$$(NA)^k = (NA) \ldots (NA) \subseteq NN \ldots NA = N^k A,$$

hence $NA \subseteq N$. This shows that N is also a right ideal. If J is any nilpotent right ideal in A, the above reasoning shows that AJ is a nilpotent right ideal, it clearly is a left ideal and so $J \subset AJ \subset N$.

<u>Definition 13</u>

The maximal two-sided nilpotent ideal N of Proposition 5 is called the <u>radical</u> of A and is denoted by rad A.

<u>Definition 14</u>

We say that A is <u>semisimple</u> if rad A = 0.

Proposition 6

The factor algebra A/rad A is semisimple.
Proof: Every left ideal in A/rad A is of the form (see Proposition 2) I/rad A for some left ideal I of A containing rad A. Then I/rad A is nilpotent in A/rad A if and only if some power of I is contained in rad A, and this can occur if and only if I is nilpotent and so is contained in rad A. Therefore A/rad A contains no nilpotent right or two sided ideals either. Hence A/rad A is semisimple.

5. Orthogonal Sets of Idempotents

Let A be a direct sum of left ideals $A = L_1 \oplus L_2 \oplus \ldots \oplus L_n$, $L_i \neq 0$. Then we may write $1 = e_1 + \ldots + e_n$, $e_i \in L_i$, for some set of elements $\{e_i\}$ of A. Clearly $e_j = e_j e_1 + . . + e_j e_n$, $e_j e_i \in L_i$. Since $L_1 \oplus \ldots \oplus L_n$ is a direct sum, this yields $e_j e_1 = 0$, $e_j e_2 = 0$, ... , $e_j e_j = e_j$, $e_j e_{j+1} = 0$, ..., $e_j e_n = 0$. Therefore

$$e_j e_i = \delta_{ij} e_i \tag{7}$$

Let us show that $L_i = A e_i$, which will imply that $e_i \neq 0$. Obviously we have $A = A e_1 + A e_2 + \ldots + A e_n$, and $A e_i \subset L_i$. This implies $A e_i = L_i$.

Definition 15

A set of idempotents $\{e_i\}$ satisfying (7) is called an orthogonal set of idempotents.

Conversely, let $\{e_i\}$ be an orthogonal set of idempotents with $e_1 + \ldots + e_n = 1$ and let $L_i = A\, e_i$, then $A = L_1 \oplus \ldots \oplus L_n$.

This is seen as follows. For an arbitrary element $a \in A$

$$a = a1 = a\, e_1 + \ldots + a\, e_n, \tag{8}$$

and hence

$$A = L_1 + \ldots + L_n. \tag{9}$$

The sum in (9) is direct, since the representation of an element as a sum $a = x_1 e_1 + \ldots + x_n e_n$ is unique. Multiplication with e_j from the right gives $ae_j = x_j e_j$.

What has been said so far in this paragraph correspondingly holds for right ideals. Hence, if we have a decomposition

$$A = L_1 \oplus \ldots \oplus L_n,\ L_i = A\, e_i, \tag{10}$$

then A is a direct sum of the right ideals $R_i = e_i A$

$$A = R_1 \oplus \ldots \oplus R_n. \tag{11}$$

If the L_i in (11) are indecomposable then the same is true for the R_i in (11).

Proof: Suppose that R_1 is decomposable, $R_1 = e_1 A = R'_1 \oplus R''_1$, then $A = e'_1 A \oplus e''_1 A \oplus e_2 A \oplus \ldots \oplus e_n A$ with $e_1 = e'_1 + e''_1$. $\{ e'_1, e''_1, e_2, \ldots , e_n \}$ is an orthogonal set of idempotents and hence $L_1 = Ae'_1 \oplus Ae''_1$, in contradiction with the assumption.

Next we consider direct decompositions of A into two-sided ideals. Let

$$A = A_1 \oplus \ldots \oplus A_n \tag{12}$$

be a decomposition of A into two-sided ideals A_i, then $A^2_i = A_i$ and $A_i A_j = 0$ for $i \neq j$.

Proof: For $i \neq j$ $A_i A_j \subset A_i \cap A_j = 0$. Furthermore, since A has a unit element $A_1 \subset A_1 A = A^2{}_1$. On the other hand $A^2{}_1 \subset A_1$, hence $A^2{}_1 = A_1$, etc.

Now assume that A is a direct sum of subspaces A_i

$$A = A_1 \oplus \ldots \oplus A_n, \qquad (13)$$

such that $A^2{}_i = A_i$ and $A_i A_j = 0$ for $i \neq j$. Then the A_i are two-sided ideals.

Proof: $AA_i = (A_1 \oplus \ldots \oplus A_n)\, A_i = A_i$, and similarly $A_i\, A = A_i$, q.e.d.

An ideal of an A_i in (13) is an ideal of A.

Proof: Let R be, for instance, a right ideal of A_1, then $RA = R(A_1 \oplus \ldots \oplus A_n) \subseteq RA_1 \subseteq R$, q.e.d.

Let R be a right ideal in A and assume a decomposition (12) of A into two-sided ideals. Then

$$R = RA = RA_1 + \ldots + R\,A_n\ .$$

This sum is direct since $RA_i \subseteq A_i$. Furthermore, $RA_i A = RA_i$, hence RA_i is a right ideal.

The last two remarks show that we know the ideals of A if we know those of the A_i. Hence we only have to investigate the ideals of indecomposable algebras.

Two right ideals which are $\neq 0$ and are contained in different components A_i in (12) cannot be isomorphic. This follows from the fact that for a right ideal $R \subset A_1$, $RA_i = 0$ for $i = 2, \ldots n$, but $RA_1 \neq 0$.

We now consider the case where the two-sided ideals in (12) are indecomposable. In this case the A_i in (12) are uniquely determined up to a permutation. Thus let $A = A'_1 \oplus \ldots \oplus A'_\ell$

be another decomposition of A into a direct sum of indecomposable two-sided ideals. Then
$A_i = A_i A = A_i A'_1 \oplus \ldots \oplus A_i A'_\ell$. Since A_i is indecomposable only one summand on the right hand side is different from 0. Hence $A_i = A_i A'_j$ for a certain j. Similarly $AA'_j = A'_j = A_i A'_j$, which implies $A_i = A'_j$. From this the proposition follows.

We now add a few remarks about the center C of A.

<u>Definition</u>
The center of A is the set of elements which commute with all elements of A.

The center C is a commutative subalgebra of A. Let us assume a decomposition (12) for A and set

$$C = C_1 \oplus \ldots \oplus C_n, \; C_i \subseteq A_i \qquad (13)$$

Then C_i is an ideal in C. To see this we choose $c \in C$ and write $c = c_1 + \ldots + c_n$, $c_i \in C_i$. From $ca = c_1 a + \ldots + c_n a = ac = a c_1 + \ldots + ac_n$, we conclude $c_i a = ac_i$, hence $c_i \in C$, or $C_i \subseteq C$.

This implies $C_i = A_i \cap C$, showing that C_i is an ideal in C.

Conversely, suppose that $C = C_1 \oplus \ldots \oplus C_n$, where the C_i are ideals in C. Let $1 = e_1 + \ldots + e_n$, $e_i \in C_i$. The $\{e_i\}$ form an orthogonal system of central ($e_i \in C$) idempotents. If we put $A_i = e_i A = Ae_i$, the A_i are obviously two-sided ideals and from earlier remarks it follows that

$$A = A_1 \oplus \ldots \oplus A_n \qquad (14)$$

We also note that $C_i = e_i C$. This follows from $C = e_1 C + \ldots + e_n C$ and $e_i C \subseteq C_i$. Hence

$$A_i = e_i A = e_i CA = C_i A \quad . \tag{15}$$

It is clear that the C_i are indecomposable if and only if the A_i are indecomposable.

6. The Structure of Semi-Simple Algebras

We recall that a left A-module M is completely reducible if every submodule of M is a direct summand of M.

Theorem 3

Let A be an algebra. If ${}_A A$ is completely reducible, then A is semisimple.

Proof: Let N be the radical of A. Since N is a left ideal of A, it is also a submodule of ${}_A A$, and therefore $A = N \oplus N'$ for some left ideal N' of A. Then $1 = e + e'$, $e \in N$, $e' \in N'$, with $e^2 = e$. Since N is nilpotent $e = e^2 = e^3 = \ldots = 0$ and therefore $N = 0$. We now prove the converse.

Theorem 4

If A is a semi-simple algebra, then ${}_A A$ is completely reducible.

Proof: Let L be a minimal left ideal. This is by definition an irreducible submodule of ${}_A A$ different from zero. Clearly $L^2 \subset L$ and $L^2 \neq 0$ (since rad $A = 0$). Hence there exists an element $a \in L$ with $L\,a \neq 0$. But $L\,a$ is a left ideal contained in L and consequently $L\,a = L$. Now consider the set

$\{ b \in L: b a = 0 \}$. This is a left ideal different from L and hence is equal to 0. Since L a = L there exists an element $c \in L$ with c a = a and hence $(c - c^2)\, a = 0$. This implies $c = c^2$ and c is an idempotent. Clearly Ac is a left ideal contained in L and thus Ac = L = Lc. An arbitrary element $x \in A$ can be written $x = xc + (x - xc)$. The set $\{ x - xc \}$ is a left ideal L' and $A = L + L'$. This sum is direct. Thus suppose that $y \in L \cap L'$, then $y = xc = xc^2 = yc$, and $y = x - xc$, hence $yc = 0 = y$. Consider $I \supset L$

$$I = A \cap I = (L \oplus L') \cap I = L \oplus (L' \cap I).$$

The last equality follows from Dedekind's modular law. So far we have shown:

Every minimal left ideal of A is generated by an idempotent element and is a direct summand of every left ideal which contains it.

Using this we proceed as follows. Let L_1 be a minimal left ideal of A. There is a L'_1 such that $A = L_1 \oplus L'_1$. If $L'_1 \neq 0$, there is a minimal left ideal L_2 in L'_1 and $L'_1 = L_2 \oplus L'_2$. Repeating this process, we obtain $A = \oplus L_i$, L_i minimal, hence ${}_A A$ is completely reducible. Theorem 3 and 4 prove the

<u>First Structure Theorem.</u>

An algebra A is semi-simple if and only if ${}_A A$ is a completely reducible left A-module.

<u>Definition 16</u>

An algebra is <u>simple</u> if the only two-sided ideals are the trivial ones.

Let us show at once that a simple algebra S is semi-simple. We note that N = rad S is a two-sided nilpotent ideal of S and that consequently N = (0) or S. The unit element cannot belong to the nilpotent ideal N, hence N = (0) and S is semi-simple.

We now prove the

<u>Second Structure Theorem.</u>

An algebra is semi-simple if and only if $A = A_1 \oplus \dots \oplus A_n$ where the A_i are two-sided ideals which are simple algebras.

Proof:
Let A be semi-simple. We show first that if B is a two-sided ideal, then there exists a two-sided ideal B' such that $A = B \oplus B'$. Since ${}_AA$ is completely reducible, there exists certainly a left ideal L of A with $A = B \oplus L$. But $BL \subseteq B \cap L = 0$, hence $(LB)^2 = LBLB = 0$. Since A is semi-simple this implies LB = 0. This means $LA = L(B \oplus L) = L^2 \subseteq L$, which proves that L is a two-sided ideal. Now let A_1 be a minimal two-sided ideal. As we have just shown, there exists a two-sided ideal A'_1 such that $A = A_1 \oplus A'_1$. We have seen earlier that every A_1 - sub-ideal is an ideal of A. Consequently, as A_1 is minimal, it is also simple. Moreover, A'_1 is clearly semi-simple. Hence, induction on dim A implies that $A'_1 = A_2 \oplus \dots \oplus A_n$, where the A_i are two-sided ideals and simple algebras. Therefore we have $A = A_1 \oplus \dots \oplus A_n$, A_i simple and two-sided ideals. Conversely, suppose that $A = A_1 \oplus \dots \oplus A_n$, where the A_i are two-sided ideals which are simple algebras. Since the ${}_{A_i}A_i$ are (by the first structure theorem) completely reducible the same is true for ${}_AA$ (give the details of the argument). This proves that ${}_AA$ is semi-simple.

Remark:

Note that the decomposition $A = \bigoplus A_i$ is unique as shown on page 15.

We study now simple algebras, since every semisimple algebra is a direct sum of simple algebras.

Let A be a simple algebra. First we show that all minimal left ideals of A are isomorphic. This follows from the following

Lemma 1

An irreducible left-module M over a simple algebra A is isomorphic to every minimal left ideal of A.

Proof: The A-module M is <u>faithful</u> ($aM = 0$, $a \in A$, implies $a = 0$). Thus let $I = \{a \in A : a M = 0\}$. Clearly I is a two-sided ideal and hence $I = 0$ or A. Since $M \neq 0$ we conclude that $I = 0$ (A has a unit element). Now let L be a minimal left ideal of A. Since M is faithful and $L \neq 0$ there exists an element $m \in M$ such that $L m \neq 0$. Since $AL \subseteq L$, the set Lm is a submodule of M and hence equal to M (M is irreducible). This shows that the mapping

$$\varphi : p \longrightarrow p\, m, \text{ for all } p \in L$$

is an A-module-epimorphism of L onto M. The kernel is an ideal in L different from L (since $Lm \neq 0$), hence equal to 0. This proves the lemma.

Since the minimal left ideals are irreducible left A-modules, we conclude that <u>all minimal left ideals of a simple algebra are isomorphic.</u> Furthermore, we see that a <u>simple algebra has precisely one irreducible module</u> (up to isomorphism).

<u>Definition:</u> A nonzero ring (with unity) is called a skew-field if every nonzero element has an inverse. A commutative skewfield is called a field.

<u>Third Structure Theorem</u>

Let A be a simple algebra. Then $A \cong \mathrm{Hom}_{\Omega}(M, M)$ for some finite-dimensional right-vector space M over a skew field Ω.

Proof:

Since ${}_AA$ is completely reducible we can decompose A into a direct sum of minimal left ideals.

$$A = L_1 \oplus \ldots \oplus L_m. \tag{16}$$

We know that the L_i are all pairwise isomorphic. Let us put $1 = e_1 + \ldots + e_m$, $e_i \in L_i$, and $R_i = e_iA$. Then we know that

$$A = R_1 \oplus \ldots \oplus R_m, \tag{17}$$

and that the R_i are minimal right ideals. Let $A_{ik} \equiv R_iL_k = e_iAL_k = e_iL_k = e_iL_ke_k = e_iA\,e_k$. Clearly

$$A = AA = \sum_{i,k} A_{ik} \quad . \tag{18}$$

The sum in (18) is direct: Consider

$$\sum_{i,k} d_{ik} = 0, \; d_{ik} \in A_{ik}.$$

Clearly $\sum_k d_{ik} = 0$ $(\sum_k d_{ik} \in R_i)$. This implies $d_{ik} = 0$ $(d_{ik} \in L_k)$. Since $Ae_kA = A$ (two-sided ideal $\neq 0$) we have

$$A_{ik}\,R_k = R_iL_kR_k = R_iL_ke_kA = R_iA\,e_k\,A = R_i\,A = R_i \tag{19}$$

Note that this implies $A_{ik} \neq 0$. Next we prove:

An A-isomorphism $\varphi : R_k \longrightarrow R_i$ can be represented as

$$\varphi\,(x_k) = a_{ik}\, x_k,\ x_k \in R_k,\ a_{ik} \in A_{ik}\ . \qquad (2o)$$

To show this we put $\varphi\,(e_k) = a_{ik} \in R_i$. From $a_{ik}\, e_k = \varphi(e_k)\, e_k = \varphi(e^2_k) = \varphi(e_k) = a_{ik}$, follows that $a_{ik} \in A_{ik}$. Now for $x_k \in R_k$, $\varphi(x_k) = \varphi(e_k x_k) = \varphi(e_k)\, x_k = a_{ik}\, x_k$. The element a_{ik} in (2o) is uniquely determined, because $\varphi(e_k) = a_{ik}\, e_k = a_{ik}$ $(a_{ik} = e_i\ a\ e_k)$. Conversely, for every $a_{ik} \in A_{ik}$, $a_{ik} \neq 0$, the mapping $x_k \longrightarrow a_{ik}\, x_k$, $x_k \in R_k$, defines an isomorphism from R_k onto R_i. These remarks show especially that the elements of A_{ii} are in one-to-one correspondence with the automorphisms of R_i

$$A_{ii} \cong \mathrm{Aut}\,(R_i), \qquad (21)$$

hence A_{ii} is a skewfield. Now let Γ_{11} be the identical automorphism of R_1, Γ_{i1} some fixed isomorphism of R_1 onto R_i and

$$\Gamma_{ik} = \Gamma_{i1}\, \Gamma_{k1}^{-1} : \ R_k \longrightarrow R_i \qquad (22)$$

Clearly

$$\Gamma_{ik}\ \Gamma_{km} = \Gamma_{im}\ . \qquad (23)$$

Let

$$\Gamma_{ik}\,(x_k) = e_{ik}\, x_k\ ,\ x_k \in R_k,\ e_{ik} \in A_{ik}. \qquad (24)$$

Then from (23) we obtain $e_{ik}\, e_{km} = e_{im}$. Furthermore, since $e_{ik} \in A_{ik}$, $e_{ij}\, e_j = e_{ij}$, $e_i\, e_{ij} = e_{ij}$. The multiplication table for the e_{ij} is the same as that of the nxn matrices which have a 1 at position (i,j) and zeros everywhere else. Now consider the mapping $A_{11} \longrightarrow A_{ii}$ given by $a_{11} \longrightarrow e_{i1} a_{11} e_{1i}$. This is a ring homomorphism. For instance $e_{i1} a_{11} b_{11} e_{1i} = e_{i1} a_{11} e_{1i} e_{i1} b_{11} e_{1i}$. The kernel of this homomorphism is zero: $e_{i1} a_{11} e_{1i} = 0$ implies $e_{1i} e_{i1} a_{11} e_{1i} e_{i1} = a_{11} = 0$.

We define

$$\Omega = \{ \alpha : \alpha = a_{11} + \dots + a_{mm} ; \quad a_{11} \in A_{11}, \ a_{ii} = e_{i1} a_{11} e_{1i} \} \tag{25}$$

It is easy to verify that the mapping $a_{11} \longrightarrow \alpha = a_{11} + \dots + a_{mm}$ is a ring isomorphism from A_{11} onto Ω. Since A_{11} is a skewfield we conclude that Ω is a skewfield. From $e_{ik} \alpha = e_{ik} a_{kk} = e_{ik} e_{k1} a_{11} e_{1k} = e_{i1} a_{11} e_{1k}$, and $\alpha e_{ik} = a_{ii} e_{ik} = e_{i1} a_{11} e_{1k}$, we conclude $\alpha e_{ik} = e_{ik} \alpha$, for all $\alpha \in \Omega$. Furthermore we note $e_{ik} \Omega = e_{ik} A_{kk} = e_{ik} R_k L_k = R_i L_k = A_{ik}$. Hence

$$A = \bigoplus_{i,k} e_{ik} \Omega . \tag{26}$$

Now we define the following mapping from A to Ω_{mxm} where Ω_{mxm} is the complete matrix algebra over the skewfield Ω

$$a = \sum_{i,k} e_{ik} \alpha_{ik} \longrightarrow (\alpha_{ik}) \in \Omega_{mxm} . \tag{27}$$

It is clear that this is an algebra isomorphism, hence

$$A \cong \Omega_{mxm} . \tag{28}$$

If A is a simple algebra over an algebraically closed field $\emptyset$, then we will show that $A \cong \emptyset_{nxn}$. It is not difficult to conclude this from the results obtained so far. We prefer however, to derive this directly. We use the same notations as in the proof of the last theorem. We have seen that for $x \in R_1$ and $0 \neq a_{11} \in A_{11}$, the mapping $x \longrightarrow a_{11}\, x$ defines an automorphism of R_1 onto R_1. Furthermore, as we have seen, two different elements $a_{11} \in A_{11}$ give different automorphisms and one gets in this way all automorphisms of R_1. The second part of Schur's lemma (Curtis-Reiner S.181) implies that every automorphism is a multiple of the unit transformation (here we use the fact that $\emptyset$ is algebraically closed). Hence, for all $x \in A$, $e_{11}\, x\, e_{11} = \xi\, e_{11}$, $\xi \in \emptyset$ $(e_{11} = e_1)$. From this we will now conclude that every $x \in A_{ik}$ is a multiple of e_{ik}. Let $x \in A_{ik}$, then $e_{1i}\, x\, e_{k1} \in A_{11}$. Hence $e_{1i}\, x\, e_{k1} = \xi\, e_{11}$. Consequently $x = e_{ii} x\, e_{kk} = e_{i1} e_{1i} x\, e_{k1} e_{1k} = e_{i1}\, \xi\, e_{11} e_{1k} =$ $= \xi\, e_{ik}$.

From $A = \bigoplus_{ik} A_{ik}$ we conclude $A = \bigoplus_{i,k} \emptyset\, e_{ik}$ and by the same reasoning as before $A \cong \emptyset_{nxn}$.

Conversely we now prove the

<u>Theorem 5</u>

Let Ω be a skewfield and let Ω_{mxm}, $m \geq 1$ be the algebra of all mxm matrices with entries in Ω, then Ω_{mxm} is a simple algebra.

Proof:

If e_{ij} denotes the matrix with 1 at position (i,j) and zeros elsewhere, the elements $\{e_{ij}: 1 \leq i,j \leq m\}$ form a basis of Ω_{mxm}. $\Omega_{mxm} = \bigoplus_{i,j} \Omega e_{ij}$. Now we take an arbitrary element $a \in \Omega_{mxm}$ different from zero and consider the two-sided ideal $I = \Omega a \Omega$ which is generated by this element. Let $a = \sum \alpha_{ik} e_{ik}$, and choose (r,s) such, that $\alpha_{rs} \neq 0$. Then

$$e_{hr} \left(\sum \alpha_{ik} e_{ik} \right) e_{sn} \alpha_{rs}^{-1} = e_{hn} \text{ ; } h, n = 1, \ldots, m$$

(by definition, if 1 is the unit element in Ω, $1 \alpha = \alpha 1 = \alpha$ for all $\alpha \in \Omega$ and thus $e_{ik} \alpha = \alpha e_{ik}$). This implies that $I = \Omega_{mxm}$, which proves that Ω_{mxm} is a simple algebra.

Let A be a simple algebra. We have seen that $A = \bigoplus_{i,k} e_{ik} \Omega$ and $\alpha e_{ik} = e_{ik} \alpha$, for all $\alpha \in \Omega$. Hence the center $C(\Omega)$ of Ω is contained in the center $C(A)$. We show that $C(\Omega) = C(A)$. Thus let $c \in C(A)$ and write $c = \sum e_{pg} \xi_{pg}$, $\xi_{pg} \in \Omega$. From $c e_{ik} = e_{ik} c$, we obtain

$$\sum_p e_{pk} \xi_{pi} = \sum_g e_{ig} \xi_{kg} . \qquad (29)$$

By multiplying this equation with e_{ik} we obtain $e_{ik} \xi_{ki} = \delta_{ik} \sum_g e_{ig} \xi_{kg}$, and hence $\xi_{ik} = 0$ for $i \neq k$. If we use this fact in (29) we find $e_{ik} \xi_{ii} = e_{ik} \xi_{kk}$ implying $\xi_{11} = \ldots = \xi_{mm} = \xi$.

The element c has thus the form $c = (e_{11} + e_{22} + \dots + e_{mm})\,\xi$. From (25) it follows that $c \in \Omega$, hence $C(A) \subseteq C(\Omega)$. For the case where A is a simple algebra over an algebraically closed field, we have $C(A) = \emptyset \cdot \mathbb{1}$.

Consider now a semi-simple algebra A over an algebraically closed field $\emptyset$. Let

$$A = A_1 \oplus \dots \oplus A_n \tag{30}$$

be the decomposition of A into simple subalgebras and $1 = e_1 + \dots + e_n$, $e_i \in A_i$, $C = C_1 \oplus \dots \oplus C_n$, $C_i \subset A_i$ where C is the center of A. We have seen earlier that $C_i = A_i \cap C$, hence $C_i = \emptyset\, e_i$. This shows that <u>the number n of simple algebras in (30) is equal to the dimension of the center C(A).</u> Furthermore, since

$$A_i \cong \emptyset_{n_i x n_i} \tag{31}$$

$$\dim A = n_1^2 + \dots + n^2_{\dim C(A)} \quad . \tag{32}$$

7. Modules for Semi-Simple Algebras

From now on all A-modules are assumed to be finite-dimensional.

Theorem 6

An algebra A is semi-simple if and only if every left A-module is completely reducible.

Proof:
If every left A-module is completely reducible, this is in particular the case for $_AA$. The first structure theorem implies that A is semi-simple.
Now let A be semi-simple and M be an A-module. We decompose

A into a direct sum of minimal left-ideals $L_i = Ae_i$, $e^2_i = e_i \neq 0$. Let $\{m_j\}$ be a basis of M. We may write

$$M = \sum_{i,j} Ae_i m_j \qquad (33)$$

Obviously each $Ae_i m_j$ is a submodule of M, but the sum in (33) need not be direct. The mapping $Ae_i \longrightarrow Ae_i m_j$ given by $ae_i \longrightarrow ae_i m_j$, $a \in A$, is clearly an A-homomorphism of Ae_i onto $Ae_i m_j$. Since $Ae_i (= L_i)$ is an irreducible A-module, the kernel of the homomorphism is either (0) or Ae_i. Hence either $Ae_i m_j$ is also irreducible, or else $Ae_i m_j = 0$. By (33) we then find M is a sum (not necessarily direct) of irreducible submodules. This implies that M is completely reducible by the following reasoning: Assume that an A-module M is a sum (not necessarily direct) of irreducible submodules M_i

$$M = M_1 + \dots + M_n \qquad (34)$$

We show that M is completely reducible. Consider $M_1 \cap M_2$. Either $M_1 \cap M_2 = 0$ or $M_1 \cap M_2 = M_1 = M_2$. In the latter case drop the term M_2 in (34) and consider $M_1 \cap M_3$. In the former case consider $(M_1 \oplus M_2) \cap M_3$. Continuing in this way we arrive at $M = M_{i_1} \oplus \dots \oplus M_{i_s}$.

In the proof of theorem 6 some summand in (33) must be different from zero, and that summand is isomorphic to Ae_i for some i. If M is assumed to be an irreducible module, then $M = Ae_i$. Hence we have

Theorem 7

If A is semi-simple, every irreducible A-module is isomorphic to some minimal left ideal of A.

This theorem says in other words, that all irreducible representations of A are contained in the left-regular representation (module ${}_A A$). Since the number of non-isomorphic left ideals is equal to the number of simple components of A, we conclude that the number of inequivalent irreducible A-modules is equal to the number of simple components of A.

If A is a semi-simple algebra over an algebraically closed field, the number of inequivalent irreducible A-modules is equal to the dimension of the center of the algebra A.

In a simple algebra $\emptyset_{nxn}$, the minimal left ideals consist of all matrices with arbitrary i th column and zeros elsewhere. To prove this we show that every element $x \neq 0$ in $L_k = \bigoplus_i \emptyset\, e_{ik}$, $\emptyset_{nxn} = \bigoplus_k L_k$ generates L_k. Thus choose in $x \in L_k$, $x = \sum_i e_{ik}\, \alpha_i \neq 0$ an j such that $\alpha_j \neq 0$. Then $\alpha_j^{-1}\, e_{\ell j} \sum_i e_{ik}\, \alpha_i = e_{\ell k}$, $\ell = 1, .., n$. Hence $\emptyset_{nxn}\, x = L_k$. Hence the dimension of the unique irreducible $\emptyset_{nxn}$- module is n.

Now let n_i, $i = 1,\ldots,N$ be the dimensions of the inequivalent irreducible A-modules for a semi-simple algebra A over an algebraically closed field $\emptyset$. Equations (31), (32) and the above remarks show that

$$\dim A = n_1^2 + \ldots + n_N^2 \qquad (35)$$
$$N = \dim C\,(A)$$

Exercise

An idempotent $e^2 = e \neq 0$ is called primitive if it cannot be expressed as a sum of two orthogonal idempotents. Let A be a semisimple algebra. Show that an idempotent $e \in A$ is primitive if and only if $A\, e$ is a minimal left ideal.

8. Finite Groups

Let G be a finite group and $\emptyset$ a field. We construct the <u>group algebra</u> $\emptyset$ G. The underlying set consists of all formal sums

$$\sum_{g \in G} \alpha_g g , \quad \alpha_g \in \emptyset \tag{36}$$

two such expressions being regarded as equal if and only if they have the same coefficients. We define operations on the formal sums by the rules

$$\sum \alpha_g g + \sum \beta_g g = \sum (\alpha_g + \beta_g) g$$

and

$$(\sum \alpha_g\, g)\,(\sum \beta_h\, h) = \sum_{g,h \in G} \alpha_g\, \beta_h\, g\, h = \sum_{r \in G} \gamma_r\, r$$

where $\gamma_r = \sum_{g \in G} \alpha_g\, \beta_{g^{-1} r}$

Finally, we define $\alpha\,(\sum \alpha_g\, g) = \sum (\alpha\alpha_g) g\,, \quad \alpha \in \emptyset$.

With these definitions, the set of all formal sums forms an algebra $\emptyset$ G, called the group algebra of G over $\emptyset$. The unit element of $\emptyset$ G is identical with that of G and is obtained from (36) by setting $\alpha(g) = 0$ for all $g \neq e$, $\alpha(e) = 1$, e = unit element of G. The formal sums $g^* = 1.g$, $g \in G$, which have all but one c oefficient equal to zero, are linearly independent and form a basis of the algebra $\emptyset$ G over $\emptyset$. The mapping $g \rightarrow g^*$ is a monomorphism of G into $\emptyset$ G, and we shall identify G with its image under this monomorphism. We can then view G as embedded in $\emptyset$ G so that the elements of G form a $\emptyset$ -basis of $\emptyset$ G.

Now let T be a representation of G with representation space M, where M is a vector space over the field $\emptyset$. Then there is a unique way to extend T to a representation T* of $\emptyset$ G with representation space M, namely

$$T^*\,(\sum \alpha_g\, g) = \sum \alpha_g\, T(g)\ .$$

Conversely, every representation of $\emptyset$ G, upon restriction to G, yields a representation of G. Moreover it is clear that a subspace $N \subset M$ is invariant under G if and only if it is invariant under $\emptyset$ G.

We always assume that $\emptyset$ has characteristic 0. The <u>charac-teristic</u> of a skewfield $\emptyset$ is defined as the order of the multiplicative unit e in the additive abelian group $\emptyset$, that is the smallest natural number m such that $m\,e = 0$.

We now show that $\emptyset$ G is semisimple. This follows from the above remarks, theorem 6 and the following

Theorem 8

Let $p: G \longrightarrow GL(M)$ be a representation of a finite group G by linear transformations on a vector space M over $\emptyset$. Then p is completely reducible.

Proof:

Let $N \neq 0$ be a G-subspace (subspace invariant under G) of M. We have to construct a G-subspace L of M such that $M = N \oplus L$. Because M is a vector space we can at least find a subspace N' (not necessary invariant under G) such that $M = N \oplus N'$.

Let $\pi: M \longrightarrow N$ be the mapping given by $m \longrightarrow n$, $m = n + n'$, $m \in M$, $n \in N$, $n' \in N'$. The projection π has the characterizing properties

$$\text{(i)}\ \pi(m) = m,\ m \in N \quad \text{(ii)}\ \pi(M) \subseteq N \tag{37}$$

In fact, for a given $\Upsilon \in Hom_{\emptyset}(M, N)$ satisfying (i) and (ii) we have the decomposition

$$M = \Upsilon(M) \oplus (1-\Upsilon)(M)$$

where $(1-\Upsilon)(M) = \{ m - \Upsilon(m) : m \in M \}$.

It is easy to see that the subspace $\Upsilon(M)$ and $(1-\Upsilon)(M)$ are G-subspaces of M if

$$p(g)\,\Upsilon = \Upsilon\,p(g), \quad g \in G. \tag{38}$$

With the help of π we now construct a projection satisfying (38). This is done by an averaging device. Let N be the number of elements of G and define

$$\tau = N^{-1} \sum_{g \in G} p\,(g)\, \pi\, p^{-1}(g). \qquad (39)$$

Then for all $h \in G$ we have, as a simple calculation shows, $p\,(h)\, \tau\, p(h)^{-1} = \tau$ and hence (38) holds. Now we note $p(g)\, \pi\, p(g)^{-1} M \subset p(g)\, \pi\, M \subset N$. Therefore by (39) $\tau(M) \subset N$, which shows that $\tau \in \mathrm{Hom}_{\emptyset}(M,N)$. It remains to show that $\tau(n) = n$ for $n \in N$. From $\pi\, p(g)^{-1} n = p\,(g)^{-1} n$, $g \in G$, $n \in N$ and (39) this follows immediately. This proves the theorem.

We also prove the

Theorem 9

The dimension of the center of $\emptyset G$ is equal to the number of conjugate classes of G.

Proof:
Let $K_1, \ldots, K_s$ denote the conjugate classes of G. For each i, define the element $c_i \in \emptyset G$

$$c_i = \sum_{g \in K_i} g, \qquad i = 1, \ldots, s.$$

The elements c_i belong to the center of $\emptyset G$ since for all $h \in G$, $h\, c_i\, h^{-1} = \sum_{g \in K_i} h\, g\, h^{-1} = c_i$. Moreover, the elements $\{c_1, \ldots, c_s\}$ are linearly independent over $\emptyset$ since they are sums of non-overlapping sets of group elements. Finally, let $x = \sum \alpha_g\, g$ belong to the center of $\emptyset G$. Then for each $h \in G$ we have $x = \sum \alpha_g\, g = h\, x\, h^{-1} = \sum \alpha_g\, h\, g\, h^{-1}$. Comparing coefficients we obtain $\alpha_g = \alpha_{h^{-1} g\, h}$ for all $g \in G$.

Thus α_g is a class function (constant on K_i), and so x is a linear combination of the c_i. This completes the proof of theorem 9.

As a corollary we obtain

Theorem 10

Let G be a finite group and $\emptyset$ an algebraically closed field (with characteristic 0). Then the number of non-isomorphic irreducible $\emptyset$ G-modules is the same as the number of conjugate classes of G.

9. The Clifford algebra

As an application of our results we discuss now the Clifford algebra which enters physics in many places. The Clifford algebra $\mathbb{C}(n)$ is the free algebra over the set $\{\gamma_1, \ldots, \gamma_n\}$ modulo the relations

$$\{\gamma_\mu, \gamma_\nu\} = 2 g_{\mu\nu} \cdot \mathbb{1} \quad . \tag{40}$$

The following set of 2^n elements γ_A forms a basis of $\mathbb{C}(n)$

$$\gamma_A : \mathbb{1} ; \ \gamma_\mu ; \ \gamma_\mu \gamma_\nu , \mu < \nu ; \ \gamma_\mu \gamma_\nu \gamma_\sigma , \mu < \nu < \sigma ; \ldots ; \gamma_1 \cdots \gamma_n .$$

Theorem 11

The Clifford algebra $\mathbb{C}(n)$ is semi-simple.

Proof:
The set of elements $\{\gamma_A , -\gamma_A : A = 1, \ldots , 2^n\}$ forms obviously a group, which generates $\mathbb{C}(n)$, Since every representation of this finite group is completely reducible, every $\mathbb{C}(n)$-module is completely reducible. By theorem 6 this proves that $\mathbb{C}(n)$ is semi-simple.

Theorem 12

The Clifford algebra $\mathbb{C}(2n)$, n = 1, 2, ... is simple.

Proof:
It is sufficient to show that the center is trivial. We show a little more. Let $X \in \mathbb{C}(2n)$ and $X\gamma_\mu = \gamma_\mu X$, then $X = \alpha \cdot \mathbb{1}$.
We write

$$X = \sum \alpha_A \gamma_A \tag{41}$$

Now we prove that for $B \neq 1$ there exists a $\mu \in \{1, \ldots, n\}$ such that

$$\gamma_\mu \gamma_B \gamma_\mu = -\gamma_B \tag{42}$$

For instance $\gamma_1 \gamma_2 \gamma_1 = -\gamma_2$, $\gamma_1 \gamma_1 \gamma_2 \cdots \gamma_{2n} \gamma_1 = - \gamma_1 \cdots \gamma_{2n}$. Now $\gamma_\mu X \gamma_\mu = (\gamma_\mu)^2 X = X$ because $X \gamma_\mu = \gamma_\mu X$, hence from (41) for a fixed $B \neq 1$ and μ satisfying (42)

$$\sum \alpha_A \gamma_\mu \gamma_A \gamma_\mu = \sum \alpha_A \gamma_A = \sum_{A \neq B} \alpha_A \gamma_A + \alpha_B \gamma_B$$

$$= \sum_{A \neq B} \pm \alpha_A \gamma_A - \alpha_B \gamma_B .$$

This implies $\alpha_B = 0$ for $B \neq 1$ and hence $X = \alpha \cdot \mathbb{1}$. As a corollary we obtain the theorem of Pauli:

Theorem 13

The Clifford algebra $\mathbb{C}(2n)$ has (up to isomorphism) precisely one irreducible representation. This representation is faithful and has dimension 2^n.

As an application of this theorem we show that the canonical anticommutation relations for a finite number of emission and absorption operators a_i, a_i^* satisfying $\{a_i, a_j^*\} = \delta_{ij}$; $i,j = 1, \ldots, N$, have only one irreducible representation. This follows from the fact that the

$$\gamma_i = a_i + a_i^* \qquad i = 1, \ldots, N$$
$$\gamma_{-i} = a_i - a_i^*$$

generate the Clifford algebra $\mathbb{C}(2N)$.

10. Irreducible Representations of the Symmetric Group.

In this paragraph we illustrate how the general theorems may be applied to construct the irreducible representations of the symmetric group.

We begin with some elementary remarks about S_n. S_n is the group of all permutations of the set $X = \{1, \ldots, n\}$. We can consider X as an S_n-set.

Definition: Let $x,y \in X$. x is S_n-equivalent to y ($x \sim y$) provided that $\sigma x = y$ for some $\sigma \in S_n$. $\sim$ is indeed an equivalence relation.

Definition: The S_n-equivalence classes of X are the orbits of X relative to S_n. We call an orbit trivial if it consists of a single element in X.

Let $[\pi]$ denote the cyclic group generated by an element $\pi \in S_n$ (this is the set of elements $\{1, \pi, \pi^2, \ldots, \pi^{g-1}\}$ where g is the smallest integer such that $\pi^g = 1$). We call π a cycle if X has only one non-trivial orbit relative to $[\pi]$.

Each cycle π acts transitively on its non-trivial orbit and we may assume without loss of generality that it cyclically permutes the elements. Hence it may be written as

$$\pi = (y, \pi y, \ldots, \pi^{g-1} y)$$

(meaning that the first object of the list takes the place of the last and each of the others replaces its left neighbour).

Two cycles are called <u>disjoint</u> if their nontrivial orbits are disjoint. It is easy to see that disjoint cycles commute with each other. Using this fact we show

<u>Lemma 2:</u>

Every permutation $\sigma \in S_n$, $\sigma \neq 1\!\!1$, is expressible as a product of disjoint cycles. This expression is unique up to order of occurrence of the factors.

Proof:

Let $X_1, \ldots X_m$ be the disjoint orbits of $[\sigma]$. Define for each i, $1 \leq i \leq m$ a cycle π_i which acts in the same way as σ on X_i and as the identity on the rest of X. (We must agree to set $\pi_i = 1$ if X_i consists of a single element, and still refer to π_i as a cycle.) We find at once that $\sigma = \pi_1 \ldots \pi_m$, a product of disjoint cycles. We remark here that $\pi \rho$ means "first ρ ; then π ". To prove the uniqueness, suppose also that $\sigma = \tau_1 \ldots \tau_q$ is a product of disjoint cycles, and let X'_i be the non-trivial orbit of τ_i. Then the X'_i give the orbits of σ hence they are just a rearrangement of the X_i. Permuting the τ_i we may assume $X'_1 = X_1, \ldots, X'_m = X_m$, $q = m$. Then for each i,

$1 \leq i \leq m$, τ_i and π_i both act as σ on X_i, and each is the identity on the complement of X_i in X. Hence $\tau_i = \pi_i$ for each i.

<u>Now we determine the conjugate classes of S_n</u>. If

$$\sigma = (x, \sigma x, \sigma^2 x, \dots , \sigma^{g-1}x)$$

is a cycle in S_n, it is easy to show that, for any $\pi \in S_n$,

$$\pi \sigma \pi^{-1} = (\pi x, \pi(\sigma x), \pi(\sigma^2 x), \dots , \pi(\sigma^{g-1}x))$$

is also a cycle in S_n of the same length as σ. Now let $\tau \in S_n$ be written as a product of disjoint cycles $\tau = \sigma_1 \dots \sigma_r$ where we put in "cycles" of length 1. Then

$$\pi \tau \pi^{-1} = (\pi \sigma_1 \pi^{-1}) \dots (\pi \sigma_r \pi^{-1})$$

gives the analogous decomposition of $\pi \tau \pi^{-1}$. This establishes

<u>Lemma 3:</u>

The cycle factorization of $\pi \tau \pi^{-1}$ is gotten from that of τ by letting π act on the digits in the cycle representation of τ.

A <u>partition</u> of n is an ordered set of integers $\{m_i\}$ satisfying $m_1 + m_2 + \dots + m_r = n$, $m_1 \geq m_2 \geq \dots \geq m_r > 0$. Each $\tau \in S_n$ gives rise to a partition of n as follows: write τ as a product of disjoint cycles σ_i of lengths m_1, m_2, ... m_r, arranged in order of decreasing length. Then lemma 3 shows that each conjugate of τ yields the same partition of n as τ does. Conversely, let $\tau' = \sigma'_1 \dots \sigma'_r$

yield the same partition of n as σ does. Then, for each i, σ_i and σ'_i have the same length, say

$$\sigma_i = (x_1, x_2, \dots x_{m_i}), \quad \sigma'_i = (x'_1, x'_2, \dots, x'_{m_i}).$$

For each i, define π on the orbit of $[\sigma_i]$ by $\pi(x_1) = x'_1, \dots, \pi(x_{m_i}) = x'_{m_i}$. Then $\pi \in S_n$, and $\pi\sigma\pi^{-1} = \sigma'$.

Consequently, we have the

<u>Proposition 7</u>

There is a one-to-one correspondence between conjugate classes in S_n and partitions of n.

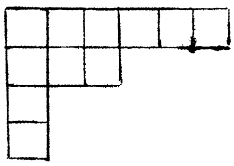
We construct now the irreducible representations of S_n. Let A be the group algebra of S_n over the field $\mathbb{C}$.

We shall determine a minimal left ideal of A corresponding to each partition in such a way that ideals which correspond to different partitions are not isomorphic (as A-modules). By Proposition 7 and Theorem 10 we can then conclude that these ideals form a complete set of non-isomorphic irreducible A-modules.

Let $\varepsilon_g = +1$ if g is an even permutation, and $\varepsilon_g = -1$ if g is an odd permutation. Let us now start with a partition $\{n_1, \dots, n_k\}$ of n,

$$n_1 + \dots + n_k = n, \quad n_1 \geq n_2 \geq \dots \geq n_k > 0.$$

With it we associate a <u>table</u> consisting of n_1 spaces in the first row, n_2 spaces in the second row, and so on.

Ex. $n = 11$, $\{6,3,1,1\} \longleftrightarrow$ 

A diagram is the array obtained by filling in the spaces of a table with the digits 1, ... , n. Starting with a diagram D, let R (D) denote the set of row permutations, that is, the set of permutations $p \in S_n$ which permute the elements in each row of D, but do not move any digit from one row to another. Then R (D) is a subgroup of S_n. Likewise define a column permutation q to be any element of S_n which permutes the elements of each column of D without moving any digit from one column to another. Let C (D) be the group of all column permutations. Obviously $R(D) \cap C(D) = (\mathbb{1})$.

The following theorem gives the main result.

Theorem 14

With each partition $\{n_1, \ldots, n_k\}$ of n we have associated a table. Each table gives rise to a collection of diagrams. For each diagram D, we obtain the group R(D) of row permutations and the group C(D) of column permutations.

If we set

$$e(D) = \sum_{\substack{p \in R(D) \\ q \in C(D)}} \varepsilon_q \, p \, q$$

then A . e(D) is a minimal left ideal in the group algebra A and thus A . e(D) is an irreducible A-module. Further, ideals coming from different diagrams with the same table are isomorphic, but ideals from diagrams with different tables are not. Hence the ideals $\{A \, . \, e(D)\}$ where D ranges over a full set of diagrams with distinct tables, gives a complete set of non-isomorphic irreducible A-modules.

In the proof of theorem 14 we shall need

Proposition 8

In the semi-simple algebra A, let L = Ae be a left ideal with generating idempotent e. Then L is minimal if and only if eAe is a skewfield.

Proof:

eAe is a non-zero subalgebra of A with unity e. Suppose that L = Ae is a minimal left ideal of A and let L_1 be any non-zero left ideal of eAe. Then $A L_1 \subset AeAe \subset Ae \subset L$ and so $A L_1$ is a left ideal of A contained in L which implies that $AL_1 = L$. But since e is a two-sided unity for eAe, we have $eAe = eL = eAL_1 = eAe\ L_1 \subset L_1$. Therefore $L_1 = eAe$, proving that the only left ideals of eAe are itself and $\{0\}$. But an algebra A' whose only left ideals are trivial is a skewfield. This can be seen as follows. Take $a \in A'$, $a \neq 0$. Suppose that there doesn't exist a $b \in A'$ with $b\,a = 1\!\!1$. Then $A'a \subset A'$ is a nontrivial left ideal, contrary to assumption. This proves that every $a \in A'$, $a \neq 0$ has an inverse and therefore A' is a skewfield.

Conversely, suppose L is not minimal. Then since every A-module is completely reducible, we may write $L = L_2 \oplus L_3$, a direct sum of non-zero left ideals of A. This gives a decomposition $e = e_2 + e_3$; $e_2 \in L_2$, $e_3 \in L_3$. We note that $e_2\,e = e_2$, $e_3\,e = e_3$ since both e_2 and e_3 lie in L. On the other hand, $e = e^2 = ee_2 + ee_3$ which shows that $ee_2 = e_2$, $ee_3 = e_3$. Therefore $e_2 = ee_2\,e$, $e_3 = ee_3\,e$, and eAe contains a pair of nonzero orthogonal elements e_2 and e_3 and hence cannot be a skewfield.
This completes the proof of proposition 8.

Proof of theorem 14:

For any diagram D and any $g \in S_n$, define gD to be the diagram obtained from D by applying g to the digits in D. Thus if α is in position (i,j) of D, then $g\alpha$ is in position (i,j) of gD.

Lemma 4

Let D' = gD, and let $h \in S_n$. If we regard hD as obtained from D by moving entries from one position to another, this same set of moves will change D' into $ghg^{-1}D'$. In other words, if the (i,j) entry of D is the (i',j') entry of hD, then the (i,j) entry of D' is the (i',j') entry of $ghg^{-1}D'$.

Proof:

If symbol α at position (i,j) in D occurs at position (i',j') in hD, the symbol β which is in position (i',j') of D must satisfy $h(\beta) = \alpha$. The element at position (i,j) in D' is of course $g(\alpha)$; that in position (i',j') of D' is $g(\beta)$. The element in (i',j') position in $ghg^{-1}D'$ is therefore

$$ghg^{-1} \cdot g(\beta) = g(\alpha) ,$$

so that, in going from D' to $ghg^{-1}D'$, the symbol $g(\alpha)$ in position (i,j) has moved to position (i',j').

Corollary. For $g \in S_n$, we have $R(gD) = gR(D)g^{-1}$, $C(gD) = gC(D)g^{-1}$, $e(gD) = ge(D)g^{-1}$.

Proof:

If $p \in R(D)$, then p leaves each entry of D in its row. By the lemma, it then follows that gpg^{-1} leaves each entry of gD in its row. This argument shows that $p \in R(D)$ if and only if $gpg^{-1} \in R(gD)$. The same holds for column permutations, and the corollary follows at once.

We use the above corollary to show that if D and D' are diagrams with the same table, then $A\cdot e(D) \cong A\cdot e(D')$. For, there exists an element $g \in \mathcal{S}_n$ such that $D' = gD$, whence $A\cdot e(D') = Age(D)g^{-1} = A\cdot e(D)g^{-1}$ since $A\cdot g = A$. But then $\theta : A\cdot e(D) \longrightarrow A\cdot e(D')$ defined by $\theta(x) = xg^{-1}$ is easily seen to be an A-isomorphism of the left A-module $A\cdot e(D)$ onto $A\cdot e(D')$.

Lemma 5

An element $g \in \mathcal{S}_n$ is expressible in the form $g = pq$, $p \in R(D)$, $q \in C(D)$, if and only if no two collinear symbols of D are co-columnar in gD.

Proof:

Assume $g = pq$, and let α, β be collinear symbols of D. Then α, β are also collinear in pD. However, $gD = (pqp^{-1})pD$, and pqp^{-1} is a column permutation for pD, so that α and β must lie in different columns of $(pqp^{-1})\ pD$.

Conversely, suppose no two collinear symbols of D are co-columnar in gD. Then any two symbols which are co-columnar in gD cannot be collinear in D. In particular, all the symbols in the first column of gD lie in different rows of D, and so there exists a row permutation $p_1 \in R(D)$ which carries all these symbols into the first column of p_1D. Repeat this procedure successively with the remaining columns of gD, thereby eventually obtaining a row permutation $p \in R(D)$ such that for each i, the i th columns of gD and pD consist of the same symbols (differently arranged, however). But then $gD = q'.pD$ for some $q' \in C(pD)$, and so $q' = pqp^{-1}$ for some $q \in C(D)$. Hence $gD = (pqp^{-1})pD = pqD$, whence $g = pq$ with $p \in R(D)$ and $q \in C(D)$. This completes the proof.

We have defined e(D) to be

$$e(D) = \sum_{\substack{p \in R(D) \\ q \in C(D)}} \varepsilon_q \, p \, q \, .$$

Remark that as p ranges over all elements of R(D) and q over C(D), the products pq thus obtained are all distinct, since $p_1q_1 = p_2q_2$ implies $p_2^{-1}\,p_1 = q_2q_1^{-1} \subset C(D) \cap R(D)$, and therefore $p_2^{-1}p_1 = q_2q_1^{-1} = (1)$. This shows that e(D) is a sum of certain group elements or their negatives, and so $e(D) \neq 0$ in A. For $p_1 \in R(D)$ and $q_1 \in C(D)$, we have at once

$$p_1e(D) = \sum \varepsilon_q \, p_1pq = e(D) \ , \tag{43}$$

$$e(D)q_1 = \sum \varepsilon_q \, pqq_1 = \varepsilon_{q_1} e(D) \quad . \tag{44}$$

We now order the partitions lexicographically, that is, if we have two partitions $n = n_1 + \ldots + n_k = n_1' + \ldots + n_h'$, $n_1 \geqslant \ldots \geqslant n_k > 0$, $n_1' \geqslant \ldots \geqslant n_h' > 0$, we write $\{n_1, \ldots, n_k\} > \{n_1', \ldots, n_h'\}$ if the first non-zero difference $n_i - n_i'$ is positive.

Lemma 6

Let D be any diagram associated with the partition $\{n_1 \ldots, n_k\}$, and D' with $\{n_1', \ldots, n_h'\}$, and suppose $\{n_1, \ldots, n_k\} > \{n_1', \ldots, n_h'\}$. Then $e(D') \cdot e(D) = 0$.

Proof:

We show first that there exist two symbols collinear in D and co-columnar in D'. Otherwise, the n_1 entries in the first row of D must occur in different columns of D'; since D' has n_1' columns, this shows that $n_1 \leqslant n_1'$, and so $n_1 = n_1'$. Now apply a column permutation on D' to obtain a new diagram D'', also associated with the partition $\{n_1', \ldots, n_h'\}$ but which has the same first row as D. We then repeat the argument with the elements of D and D'' not in the first row, getting $n_2 = n_2'$, ..., which is impossible.

We have thus shown the existence of symbols α, β collinear in D and co-columnar in D'. Set $t = (\alpha\beta) \in S_n$. Then $t \in R(D)$, $t \in C(D')$, and

$$e(D') \cdot e(D) = e(D')t \cdot te(D) = - e(D') \cdot e(D)$$

by formulas (43) and (44). Therefore $e(D')e(D) = 0$, and Lemma 6 is proved.

We note that, for $p \in R(D)$, $q \in C(D)$, $\gamma \in \mathbb{C}$, we have $p \cdot \gamma e(D) \cdot q = \varepsilon_q \cdot \gamma e(D)$.

We now prove, conversely, that the above property characterizes the scalar multiples of $e(D)$.

Lemma 7

Let $x \in A$ be such that $pxq = \varepsilon_q x$ for all $p \in R(D)$, $q \in C(D)$. Then there exists $\gamma \in \mathbb{C}$ such that $x = \gamma e(D)$.

Proof:

Let $x = \sum \alpha_g g$, the sum extending over all $g \in S_n$, where each $\alpha_g \in \mathbb{C}$. Then

$$x = \varepsilon_q p^{-1}xq^{-1} = \varepsilon_q \sum_g \alpha_g(p^{-1}gq^{-1}) = \varepsilon_q \sum_h \alpha_{phq} h$$

for each $p \in R(D)$, $q \in C(D)$. Thus

$$\alpha_g = \varepsilon_q \alpha_{pgq}, \quad p \in R(D),\ q \in C(D). \tag{45}$$

Setting $g = (\mathbb{1})$, we obtain $\alpha_{pq} = \varepsilon_q \alpha_{(\mathbb{1})}$, $p \in R(D)$, $q \in C(D)$.
To complete the proof of the lemma, we need only show that $\alpha_g = 0$ whenever g is not of the form pq for $p \in R(D)$, $q \in C(D)$. Suppose that g is not of this form; by Lemma 5, there must then exist symbols α, β collinear in D and co-columnar in gD. Let $t = (\alpha\beta) \in S_n$; then $t \in R(D)$, $t \in C(gD)$, and so $t = gqg^{-1}$ for some $q \in C(D)$. Then q is also a transposition, and we have from (45) $\alpha_g = \alpha_{tgq^{-1}} = \varepsilon_{q^{-1}} \cdot \alpha_g = -\alpha_g$ since $tgq^{-1} = g$. Therefore $\alpha_g = 0$, which proves the lemma.

Now we have for $p \in R(D)$ and $q \in C(D)$,

$$p \circ e(D)^2 \cdot q = pe(D) \circ e(D)q = \varepsilon_q e(D)^2$$

[using (44) and (45)]. By the preceding lemma, we then have $e(D)^2 = \gamma e(D)$ where γ is the coefficient of $\mathbb{1}$ in $e(D)^2$, and hence is an integer. We shall show that $\gamma \neq 0$. Let $T \in \mathrm{Hom}_{\mathbb{C}}(A,A)$ be defined by $T(x) = x \circ e(D)$, $x \in A$, and let us consider the matrix description of T obtained by using the $\mathbb{C}$-basis of A consisting of the elements $g_1 = (\mathbb{1})$, g_2, ..., $g_{n!}$ of S_n. Then if $e(D) = \alpha_1 g_1 + \dots$, $\alpha_1 \in \mathbb{C}$, we have

$$g_1 \circ e(D) = \alpha_1 g_1 + \dots$$
$$g_2 \circ e(D) = * + \alpha_1 g_2 + \dots$$

so that the trace of the matrix describing T is $\alpha_1 n!$.

Furthermore $\alpha_1 = 1$, since $(1\!\!1)$ occurs with coefficient 1 in e(D).

On the other hand, let us calculate the trace using a different $\mathbb{C}$-basis of A. We must get the same result, since the trace is independent of the basis used. Let $\{v_1, \ldots, v_{n!}\}$ be a $\mathbb{C}$-basis of A such that $\{v_1, \ldots, v_f\}$ is a $\mathbb{C}$-basis of $A \circ e(D)$. Here, $f = \dim A \circ e(D) \geq 1$ since e(D) is a non-zero element of $A \circ e(D)$. Further, $x \circ e(D) = \gamma x$ for $x \in A \circ e(D)$, and so

$$\begin{aligned} v_1 \circ e(D) &= \gamma v_1 \\ v_2 \circ e(D) &= \quad \gamma v_2 \\ &\ldots \\ v_f \circ e(D) &= \qquad \gamma v_f \\ v_{f+1} \cdot e(D) &= * + \ldots + * + 0 \\ &\ldots \\ v_{n!} \cdot e(D) &= * + \ldots + * + 0, \end{aligned}$$

since $y \circ e(D) \in A \circ e(D)$ for $y = v_{f+1}, \ldots, v_{n!}$. The trace is thus γf, so we have $\gamma f = n!$, whence $\gamma \neq 0$.

We may now show that each ideal $A \circ e(D)$ is minimal. Let $u = \gamma^{-1} e(D)$, so that $u^2 = u \neq 0$. Then u is idempotent, and

$$Au = A \circ e(D), \quad uAu = e(D)Ae(D).$$

In order to show that Au is a minimal left ideal of A, it suffices [by Proposition8] to prove that uAu is a skewfield. Let $x \in uAu$; then $x = e(D)y\, e(D)$ for some $y \in A$, and so $pxq = pe(D) \circ y \circ e(D)q = e(D)y\, e(D)\, \varepsilon_q = \varepsilon_q x$ for all $p \in R(D)$, $q \in C(D)$. By Lemma 7, x must therefore be a scalar multiple of e(D). Thus $uAu = \mathbb{C} u \cong \mathbb{C}$, which shows in fact that uAu is a field.

Finally, let D, D' be diagrams with different tables, and let Au, Au' be the minimal left ideals associated with them. We shall show that Au and Au' are not isomorphic.

Assume $\varphi : Au \longrightarrow Au'$ to be an A-isomorphism. Then $\varphi(a u) = a \varphi(u)$ for every $a \in A$ and hence also for $a \in Au$. But $a u = a$ when $a \in Au$, so we have $\varphi(a) = a \varphi(u)$, $a \in Au$. Setting $a' = \varphi(u) \in Au'$, we have $Au' = Au \cdot a'$. Hence $u = bu'a'$ for some $b \in A$, so that $u = u^2 = bu'a'u$. We shall show that $u'a'u = 0$, which will give the desired contradiction. It suffices to prove that $u'g u = 0$ for all $g \in S_n$. However, $u'g u = u'g u g^{-1}g$, and $u' g u g^{-1} = 0$ by Lemma 6 since u' and $g u g^{-1}$ come from D' and gD, respectively.

This completes the proof of Theorem 14.

CHAPTER II CENTRALIZERS OF MODULES OVER SYMMETRIC ALGEBRAS

1. Trace Form of a Semi-Simple Algebra

Let a be any element of an algebra A over a field $\emptyset$. The left multiplication of A, L_a, which is determined by a is defined by $L_a: x \rightarrow ax$, $x \in A$. Clearly L_a is a linear operator in A and $a \longrightarrow L_a$ is a representation (left regular representation) of A. We prove now the

Theorem 1

If A is a semi-simple algebra, then the trace form

$$f(x, y) = \mathrm{Tr}\,(L_x L_y) \qquad (1)$$

is a non-degenerate bilinear form which is symmetric:

$$f(x, y) = f(y, x) \qquad (2)$$

and associative:

$$f(xy, z) = f(x, yz) \quad . \qquad (3)$$

Proof:

The form f is obviously bilinear and symmetric. From

$$L_{xy} L_z = L_x L_{yz}$$

it follows that f is associative. It remains to be shown that f is non-degenerate. Let B be a two-sided ideal in A and consider $B^{\perp} = \{ y : f(x,y) = 0 \text{ for all } x \in B \}$. For $x \in B$, $y \in B^{\perp}$ and any $a \in A$, $f(x, ay) = f(xa,y) = 0$ since $xa \in B$. Also $f(x,ya) = f(ya,x) = f(y,ax) = 0$ since $a x \in B$. Hence $B^{\perp}$ is an ideal. Now assume that $A^{\perp} \neq 0$. Since A is semi-simple $A^{\perp}$ is a direct summand $A = A^{\perp} \oplus A'$ where A' is a two-sided ideal. We write $1 = e + e'$, $e \in A^{\perp}$, $e' \in A'$. From $e^2 = e \neq 0$ we obtain $0 = f(e,e) = \mathrm{Tr}\,(L^2_e) = \mathrm{Tr}\,L_e \neq 0$ [L_e is a projection operator on $A^{\perp}$]. From this contradiction we conclude $A^{\perp} = 0$ and hence that f is non-degenerate. Theorem 1 shows that a semi-simple algebra is a symmetric algebra in the sense of the following

Definition 1

A (finite-dimensional) algebra A over a field $\emptyset$ is called a symmetric algebra if A admits a non-degenerate bilinear form $f : A \times A \longrightarrow \emptyset$ which is symmetric and associative.

In this chapter we denote by A a symmetric algebra over a field $\emptyset$, M a left A-module, and $C = Hom_A\ (M; M)$. We shall show a reciprocity between certain right ideals of A and the C-submodules of M. These results hold in particular if $A = \emptyset G$ is the group algebra of a finite group, In the next chapter we shall apply these results to construct the irreducible tensor representations of the full linear group and certain of its subgroups. We follow very closely the treatment in C. Curtis and I. Reiner, Representation Theory of Finite Groups and Associative Algebras, page 440.

Let now A be a symmetric algebra with associative form f. Let $\{a_i\}$ and $\{b_j\}$ be dual bases of A with respect to f: $f(a_i, b_j) = \delta_{ij}$.

Because of the symmetry of f, we have the following two sets of relations

$$a_i\ a = \sum_j \lambda_{ij}\ (a)\ a_j \Longleftrightarrow a\ b_i = \sum_j \lambda_{ji}\ (a)\ b_j$$

and (4)

$$b_i\ a = \sum_j \mu_{ij}\ (a)\ b_j \Longleftrightarrow a\ a_i = \sum_j \mu_{ji}\ (a)\ a_j\ .$$

As an illustration, we prove the first of these two relations. Assume that $a_i\ a = \sum_j \lambda_{ij}\ (a)\ a_j$ and expand $a\ b_i = \sum_j \xi_{ji}\ b_j$.

Then $\xi_{ji} = f(a_j, a\, b_i) = f(a_j\, a, b_i) = \lambda_{ji}(a)$.

Now let M be a left A-module, and let $M^* = \mathrm{Hom}_\emptyset(M,\emptyset)$ be the $\emptyset$-dual of M. By the definition

$$(\psi a)(x) = \psi(a\, x), \quad \psi \in M^*, \quad x \in M, \quad a \in A \qquad (5)$$

M^* becomes a right A-module. We define a mapping Υ: $M \times M^* \to A$ by the rule

$$\Upsilon(x,\psi) = \sum_i b_i\, \psi(a_i\, x) \quad . \qquad (6)$$

The mapping Υ is obviously biadditive. Using (4) we furthermore show that Υ is A-bilinear in the sense that

$$\Upsilon(a\, x + b\, y, \psi) = a\, \Upsilon(x,\psi) + b\, \Upsilon(y,\psi)$$

and (7)

$$\Upsilon(x, \varphi\, a + \psi b) = \Upsilon(x, \varphi)\, a + \Upsilon(x,\psi)\, b \quad .$$

Consider for instance

$$\Upsilon(a\, x + b\, y, \psi) = \sum_i b_i\, \psi(a_i\,(a\, x + b\, y))$$

$$= \sum_i b_i\,(\psi(a_i a x) + \psi(a_i b y))$$

$$= \sum_i b_i\, \psi(a_i a x) + \sum_i b_i\, \psi(a_i b y) \quad .$$

The first term after the last equality sign is, using (4)

$$\sum_i b_i \psi(a_i a x) = \sum_i b_i \psi\Big(\sum_j \lambda_{ij}(a)\, a_j x\Big)$$

$$= \sum_{ij} b_i\, \lambda_{ij}(a)\, \psi(a_j x) = \sum_j a\, b_j\, \psi(a_j x) = a\, \Upsilon(x,\psi).$$

In the same way one treats the second term, proving the first equation in (7). Similarly one verifies the second relation in (7).

The function τ is non-degenerate. Suppose for example that $\tau(x,\psi) = 0$ for all $x \in M$; since the $\{b_i\}$ are linearly independent, we then have $\psi(a_i x) = 0$, and since $\mathbb{1}$ is a linear combination of the $\{a_i\}$, we have $\psi(M) = 0$ and $\psi = 0$. Similarly $\tau(x,\psi) = 0$ for all ψ implies $x = 0$. Because of the A-bilinearity of the form τ, the set A_M

$$A_M = \left\{ \sum_{i=1}^{n} \tau(x_i, \psi_i) : x_i \in M, \ \psi_i \in M^*, \ n \text{ finite} \right\} \quad (8)$$

is a two-sided ideal in A called the <u>nucleus of M</u>.

From now on we assume that A is semi-simple (one could keep the discussion more general). In this case the nucleus A_M contains an idempotent ε such that $a\varepsilon = \varepsilon a = a$ for all $a \in A_M$. This follows from the fact that $A = A_M \oplus A'$ for some two-sided ideal A'. If we put $\mathbb{1} = \varepsilon + \varepsilon'$, $\varepsilon \in A_M$, $\varepsilon' \in A'$, then $A_M = \varepsilon A = A\varepsilon$ and ε is a central idempotent. The element ε can be expressed in the form

$$\varepsilon = \sum \tau(x_i^0, \psi_i^0), \quad x_i^0 \in M, \ \psi_i^0 \in M^*. \quad (9)$$

For all $\psi \in M^*$ we have $\tau(\varepsilon x - x, \psi) = \varepsilon\tau(x,\psi) - \tau(x,\psi) = 0$. Hence $\varepsilon x = x$ (τ is non-degenerate). Similarly $\psi\varepsilon = \psi$ for all $\psi \in M^*$.

Let $C = \mathrm{Hom}_A(M,M)$. Our object is to study the properties of M as a C-module. We first show that the A-module M has the "double centralizer" property.

<u>Theorem 2</u>

Let $\gamma \in \mathrm{Hom}_C(M,M)$. Then there exists an element $a \in A_M$ such that $ax = \gamma x$ for all $x \in M$.

Proof:

For each pair $(\psi, x) \in M^* \times M$ we construct the endomorphism $\psi \,\square\, x$ of M by the rule

$$(\psi \,\square\, x)\, m = \tau(m, \psi)\, x \;, \quad m \in M \;. \qquad (10)$$

We show that $\psi \,\square\, x \in C$. For $a \in A$

$$(\psi \,\square\, x)\,(a\, m) = \tau(a\, m, \psi)\, x = a(\tau(m, \psi)\, x) = a((\psi \,\square\, x) m).$$

The given $\gamma \in Hom_C\,(M,M)$ must commute with all the endomorphisms $\psi \,\square\, x$, and we have for all $m \in M$, $(\psi \,\square\, x)(\gamma m) = \gamma((\psi \,\square\, x)\, m)$ or, inserting the definition (10)

$$\tau(\gamma m, \psi)\, x = \gamma(\tau(m, \psi)\, x) \qquad (11)$$

for all $m, x \in M$ and $\psi \in M^*$. Now let $a = \sum \tau(\gamma x_i^0, \psi_i^0) \in A_M$. Using (11), we obtain for all $m \in M$ $a\, m = \sum \tau(\gamma x_i^0, \psi_i^0)\, m = \gamma(\varepsilon m) = \gamma m$. This proves the theorem.

□

We establish now a connection between right ideals in A_M and C-submodules of M. For each right ideal $I \subset A_M$, $I\, M$ is a C-submodule of M. If $N \subset M$, then

$$\tau(N, M^*) = \left\{ \sum \tau(n_i, \psi_i) \;:\; n_i \in N \;,\; \psi_i \in M^* \right\} \qquad (12)$$

is a right ideal of A in A_M, because of the bilinearity of τ. In particular $\tau(M, M^*) = A_M$. Let now N be a C-submodule of M. The mappings $I \longrightarrow IM$ and $N \longrightarrow \tau(N, M^*)$ have the following properties

$$I \supset \tau(I\, M, M^*) \quad \text{and} \quad N \supset \tau(N, M^*)\, M \qquad (13)$$

the first since $\tau(I\, M, M^*) = I\, \tau(M, M^*) \subset I$ and the second because for all $n \in N$, $\psi \in M^*$, $x \in M$, $\tau(n, \psi)\, x = (\psi \,\square\, x)\, n$ and N is a C-submodule of M. We are interested in the case when these inclusions can be replaced by equality. We shall call an A-direct summand of the A-module M an A-component. First we proof

Lemma 1:

Let N be a C-component of M. Then there exists an idempotent $e \in A_M$ such that $\Upsilon(N, M^*) = e\,A$ and $\Upsilon(N, M^*)\,M = N$.

Proof:

Let π be a projection of M onto N. From $\pi \in \mathrm{Hom}_C\,(M,M)$ and the proof of Theorem 2 $\pi\,x = \sum \Upsilon(\pi\,x_i^o, \psi_i^o)\,x = e\,x$, $x \in M$ where $e = \sum \Upsilon(\pi\,x_i^o, \psi_i^o) \in \Upsilon(N,M^*)$. If $a = \sum \Upsilon(n_i, \psi_i)$ is an arbitrary element of $\Upsilon(N,M^*)$ then $e\,a = \sum \Upsilon(e\,n_i, \psi_i) = a$ since $e\,n_i = \pi\,n_i = n_i$ for all $n_i \in N$. Therefore, $\Upsilon(N,M^*) = e\,A$ and $e^2 = e$. Now let $x \in N$; then $x = e\,x \in \Upsilon(N,M^*)\,M$. In view of (13), this implies $N = \Upsilon(N,M^*)\,M$, and the Lemma is proved.

Lemma 2:

Let $I = eA$ where $e^2 = e \in A_M$. Then $IM = eM$ is a C-component of M, and $\Upsilon(IM, M^*) = I$.

Proof:

We have $IM = eAM = eM$ and $M = eM \oplus (1-e)M$, where eM and (1-e)M are C-submodules of M, thus proving the first statement. For the second, let $a \in I$; then

$$a = a\,\varepsilon = \sum \Upsilon(a\,x_i^o, \psi_i^o) \in \Upsilon(I\,M, M^*)\,.$$

This result combined with (13) proves the second statement, and the Lemma is established. □

Because A is assumed to be semi-simple, the two-sided ideal A_M is semi-simple and hence is a completely reducible right A_M-module. We show now that M is a completely reducible C-module.

Lemma 3:

Let A be a semi-simple algebra and M a left A-module. Let $C = \mathrm{Hom}_A(M,M)$. Then M is a completely reducible C-module. If e is a primitive idempotent in A_M then eM is an irreducible C-submodule of M.

Proof:

We prove first the second statement. Suppose that N is a non-zero C-submodule of eM. Then from Lemma 2 we obtain $e\,A = \Upsilon(eM,M^*) \supset \Upsilon(N,M^*)$ and since $eA = eA_M$ is a minimal ideal in A_M and $\Upsilon(N,M^*) \neq 0$ we have $eA = \Upsilon(N,M^*)$. Then by (13) $\Upsilon(N,M^*)\,M = e\,M \subset N$, and we have shown that eM is an irreducible C-module. Because A_M is a completely reducible right A_M-module, we have $A_M = \sum eA_M$, where the $\{eA_M\}$ are minimal right ideals. It follows that ($\varepsilon \in A_M$ is the unit operator in M) $M = A_M\,M = \sum e\,M$ where the $\{eM\}$ are irreducible C-modules. This proves that M is a completely reducible C-module.

Now we come to the main theorem of this chapter.

Theorem 3

Let A be a semi-simple algebra and M a left A-module. Let $C = \mathrm{Hom}_A(M,M)$. Then $I \rightarrow IM$ is a one-to-one mapping of the set of all right ideals of A_M onto the set of all C-submodules of M. Every C-submodule has the form eM where e is an idempotent in A_M. The C-module eM is irreducible if and only if e is a primitive idempotent in A_M. Two right ideals $I_i = e_i\,A$, $i = 1, 2$, generated by idempotents belonging to A_M, are A-isomorphic if and only if the C-modules e_iM, $i = 1, 2$, are C-isomorphic.

<u>Proof:</u>

From the Lemmas (1), (2) and (3), we know that the mappings $I = e A \longrightarrow I M = e M$ and $N \longrightarrow \tau(N,M^*)$ are one-to-one and inverses of each other. From Lemma 3 we know that eM is irreducible if and only if e is primitive (the "only if" part is trivial). It remains to prove the last statement of the theorem. Let $\theta : e_1 A \rightarrow e_2 A$ be an A-isomorphism, and let $\theta(e_1) = a$, $\theta^{-1}(e_2) = b$. Then $\theta(e_1 c) = \theta(e_1)\, c = \theta(e_1) e_1 c = a\, e_1 c$, $c \in A$ and $\theta^{-1}(e_2 d) = \theta^{-1}(e_2) d = b\, e_2 d$, $d \in A$. Therefore $e_2 A = a\, e_1 A$, $e_1 A = b\, e_2 A$. Moreover, $bac = c$ for all $c \in e_1 A$, and $abd = d$ for all $d \in e_2 A$. To prove the last statement, consider $c = e_1 c \in e_1 A$ and $\theta^{-1} \circ \theta(c) = c = \theta^{-1}(a\, e_1 c) = \theta^{-1}(e_2 a\, c) = b\, a\, c$. Now we define mappings $\tilde{\theta}$ and $\tilde{\xi}$ between $e_1 M$ and $e_2 M$ by the rules $\tilde{\theta}(e_1 x) = a\, e_1 x$ and $\tilde{\xi}(e_2 y) = b \cdot e_2 y$. It is clear that $\tilde{\theta}$ and $\tilde{\xi}$ are C-homomorphisms. Furthermore $\tilde{\xi} \circ \tilde{\theta}(e_1 x) = b\, a\, e_1 x = e_1 x$ and hence $\tilde{\xi} \circ \tilde{\theta} = 1$; similarly $\tilde{\theta} \circ \tilde{\xi} = 1$. This proves that $e_1 M$ and $e_2 M$ are C-isomorphic.

Conversely, suppose we have a C-isomorphism $\eta : e_1 M \longrightarrow e_2 M$. Define $\bar{\eta} : \sum \tau(x_i, \psi_i) \longrightarrow \sum \tau(\eta x_i, \psi_i)$ for $x_i \in e_1 M$, $\psi_i \in M^*$. We know that $\tau(e_1 M, M^*) = e_1 A$, $\tau(e_2 M, M^*) = e_2 A$ (see the proof of Lemma 3) and $\bar{\eta}$ maps $e_1 A$ onto $e_2 A$. We prove next that $\bar{\eta}$ is well defined. For this it is sufficient to show that $\sum \tau(x_i, \psi_i) = 0$ implies $\sum \tau(\eta x_i, \psi_i) = 0$. If $\sum \tau(x_i, \psi_i) = 0$ for $x_i \in e_1 M$, $\psi_i \in M^*$, then $0 = \sum \tau(x_i, \psi_i) M = \sum (\psi_i \square M)\, x_i$ and since η is a C-isomorphism (remember that $\psi \square x$ is in C)

$$0 = \sum \eta(\psi_i \square M)\, x_i = \sum \tau(\eta x_i, \psi_i) M . \quad (14)$$

Since $\varepsilon = \sum \tau(x_i^0, \psi_i^0)$ is a right identity element in A_M, we have by (14)

$$\sum \tau(\eta x_i, \psi_i) = \sum \tau(\eta x_i, \psi_i)\, \varepsilon = 0 \text{ by } (14)$$

and we have proved that $\bar{\zeta}$ is well defined. A similar argument shows that

$$\bar{\theta} : \sum \tau(x_i, \psi_i) \longrightarrow \sum \tau(\zeta^{-1}x_i, \psi_i)$$

for $x_i \in e_2 M$, $\psi_i \in M^*$, is a well defined mapping of $\tau(e_2M, M^*)$ onto $\tau(e_1M, M^*)$ such that $\bar{\zeta} \circ \bar{\theta} = 1$, $\bar{\theta} \circ \bar{\zeta} = 1$.

Moreover, it is clear that the mappings $\bar{\theta}$ and $\bar{\zeta}$ are right A-homomorphisms between the right ideals $\tau(e_2M, M^*)$ and $\tau(e_1M, M^*)$. It follows that $\tau(e_2M, M^*) \cong \tau(e_1M, M^*)$ as right A-modules. This completes the proof of Theorem 3. $\square$

In applications of our result it may be easy to find primitive idempotents in A but difficult to tell whether they belong to the ideal A_M. Because of this problem, the following result will be useful.

<u>Corollary</u>

Let A and M satisfy the hypothesis of Theorem 3. Let e be any primitive idempotent in A. Then either eM = 0 or eM is an irreducible C-submodule of M.

<u>Proof:</u>

We know that there is an idempotent $\varepsilon \in A_M$ such that $\varepsilon a = a\varepsilon = a$ for all $a \in A_M$, and $\varepsilon m = m$ for all $m \in M$. Let $B = \{ b \in A : bM = 0 \}$.

Then, since $\varepsilon = \sum \tau(x_i^0, \psi_i^0)$, $b \in B \cap A_M$ implies $b = b\varepsilon = \sum \tau(bx_i^0, \psi_i^0) = 0$. For all $a \in A$, $a\varepsilon - \varepsilon a \in B \cap A_M$ and it follows that ε is a <u>central</u> idempotent in A. If e is any primitive idempotent in A, $e = e\varepsilon + e(1 - \varepsilon)$ where $e\varepsilon$ and $e(1 - \varepsilon)$ are orthogonal. Since e is a primitive idempotent, either $e\varepsilon = e$ and $e \in A_M$ so that by Theorem 3, eM is an irreducible C-module, or $e(1 - \varepsilon) = e$ and eM = 0. This completes the proof of the Corollary. $\square$

APPENDIX TO CHAPTER II

In the following Appendix we add a few remarks which make the results of Chapter II a bit more explicit.

We consider the situation of Chapter II, namely a semi-simple algebra A, a left A-module M and the centralizer C. M can be considered as an (A,C)-module.

Now let B be a two-sided simple ideal in the nucleus A_M and let $M_B = BM$. Clearly M_B is invariant under C. Since $AB \subseteq B$, M_B is also invariant under A. This means that M_B is an (A,C)-submodule. We now show that M_B is an irreducible (A,C)-submodule. Assume that N is an (A,C)-submodule of M_B. Since we have a decomposition $A = B \oplus B'$ where B' is a two sided ideal, we obtain (since $B\,B' = 0$) $N = AN = BN$. Let $B = \oplus\ e_i A$ be the decomposition of B into minimal right ideals. Then $N = \sum e_i N$. Since N is by assumption $\neq 0$, there exists an i with $e_i N \neq 0$ and hence $x \in N$ with $e_i x \neq 0$. Now $C\,e_i x = e_i Cx$ is a C-submodule of $e_i M$, and since $e_i M$ is irreducible, we have $N \supseteq e_i M$. Then, however, ($A\,e_i A$ is a two-sided ideal in B, hence equal to B)
$N = AN \supseteq A\,e_i M = A\,e_i AM = BM = M_B$. This proves our statement.

We can decompose M_B into irreducible C-submodules by

$$M_B = \oplus\ e_i\, M.$$

On the other hand we can decompose M_B into irreducible A-modules

$$M_B = \oplus\ M_i\ .$$

Each M_i is isomorphic to a minimal left-ideal of A. We now show, that M_i is isomorphic to a minimal left ideal in B. Otherwise (see Schur's Lemma) for every minimal left ideal L in B, the homomorphism of L into M_i : $\ell \longrightarrow \ell x$, $\ell \in L$, $x \in M_i$, where x is fixed but arbitrary, would be the zero map. This would imply $BM_i = 0$. On the other hand $M_i = AM_i = BM_i = 0$. This contradiction proves our claim.

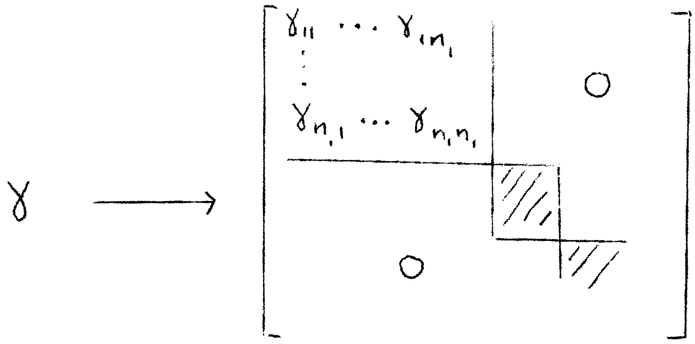
For later applications, we make this result more explicit. To do this we consider the following situation. Let M be a vector space over an algebraically closed field $\emptyset$. Let Σ be a set of linear transformations of M. We can consider M as a Σ-module. We assume that M is a completely reducible Σ-module. Next we consider the centralizer $C = \mathrm{Hom}_{\Sigma}(M,M)$ and study this centralizer. The following considerations are independent of our previous results. Let $M = \bigoplus M_j$ be the decomposition of M into irreducible Σ-modules. This decomposition defines in the usual way projections π_j of M onto M_j. The π_j are Σ-module homomorphisms and $\sum \pi_j = 1$. For a $\gamma \in C$ we put

$$\gamma = 1 \circ \gamma \circ 1 = \sum \gamma_{ij} \qquad (15)$$

where $\gamma_{ij} = \pi_i \circ \gamma \circ \pi_j$. The γ_{ij} are Σ-homomorphism of M_j onto M_i. If $\gamma \in C$ varies over C the γ_{ij} vary over all of $\mathrm{Hom}_{\Sigma}(M_j,M_i)$ since each sum (15) with arbitrary $\gamma_{ij} \in \mathrm{Hom}_{\Sigma}(M_j,M_i)$ is an element of C. The decomposition (15) is unique: Multiplication of (15) with π_i from the left and with π_j from the right gives necessarily $\gamma_{ij} = \pi_i \circ \gamma \circ \pi_j$. For a product of two γ and γ' the γ_{ij} multiply like matrices

$$\gamma \gamma' = \sum_{h,k} \left(\sum_j \gamma_{hj} \gamma'_{jk} \right) .$$

Now we collect the M_i into isomorphic sets $(M_1, \dots, M_{n_1})$, $(M_{n_1+1}, \dots, M_{n_2})$, etc. Obviously $\gamma_{ij} = 0$ if M_i is not isomorphic to M_j. For isomorphic M_i and M_j, γ_{ij} is an isomorphism. These remarks show that γ has block form



Now we consider one block $(M_1, \ldots, M_{n_1})$. We introduce isomorphic bases in the M_i.

$$\begin{array}{lllll} M_1 & : & u_{11} & \cdots & u_{1r} \\ M_2 & : & u_{21} & \cdots & u_{2r} \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ M_{n_1} & : & u_{n_1 1} & \cdots & u_{n_1 r} \end{array} \qquad (16)$$

The second part of Schur's Lemma implies that in these bases

$$\gamma_{ij} = \lambda_{ij} \mathbb{1} \quad , \quad \lambda_{ij} \in \emptyset \, .$$

If we restrict γ to $M_1 \oplus \ldots \oplus M_{n_1}$ we have

$$\gamma u_{k\ell} = \sum_{ij} \gamma_{ij} u_{k\ell} = \sum_{ij} \lambda_{ij} \delta_{kj} u_{i\ell} = \sum_{i} \lambda_{ik} u_{i\ell} \quad .$$

This shows that the columns $u_{1\ell} \ldots u_{n\ell}$ in (16) transform in the same way (independent of ℓ). To summarize, we have established that the rows in (16) transform irreducibly and in the same way under $\sum$ and that the columns transform irreducibly and in the same way under C (irreducibly since every transformation which transforms the columns in the same way belongs to C). We also see that the centralizer C is a direct sum of full matrix algebras.

These results can be applied to the modules M_B introduced at the beginning of this Appendix. They imply that we can introduce a basis of the type (16) in M_B such that the rows transform in the same way under A and the columns transform in the same way under C (always irreducibly). Note also that in M_B we have collected all isomorphic irreducible A-modules and all isomorphic irreducible C-modules of M. All this is in particular true for $A = \emptyset S_f$ and $M = V^{\otimes f}$.

A further application of the above results is the following one. Let M be an irreducible module for the direct product $G_1 \times G_2$ of the groups G_1 and G_2 and assume that M is completely reducible with respect to the group G_1 (We identify $G_1 \times 1\!\!1_2$ with G_1, etc). This is certainly the case if G_1 is finite or compact. Since $(g_1, 1)(1, g_2) = (1, g_2)(g_1, 1)$ the group G_2 operates by elements in the centralizer $Hom_{G_1}(M,M)$. For this reason G_2 leaves a "block" invariant. If we introduce a basis in M as in (16), then the rows transform irreducibly and in the same way under G_1 while the columns transform irreducibly and in the same way under G_2 (irreducibly since otherwise M would not be an irreducible $G_1 \times G_2$ -module). This proves that M is isomorphic to $M_1 \otimes M_2$ where M_i is an irreducible G_i-module and with the following operation of $G_1 \times G_2$

$$(g_1, g_2) : x_1 \otimes x_2 \longrightarrow g_1 x_1 \otimes g_2 x_2 \quad .$$

One can show conversely that a $G_1 \times G_2$-module of this form is irreducible. (Prove this by showing that $Hom_{G_1 \times G_2}(M_1 \otimes M_2, M_1 \otimes M_2) = \emptyset \cdot 1\!\!1$).